SETS OF POINTS THAT ARE THE VERTICES OF

K NON-OVERLAPPING N-GONS

Erich Friedman
Stetson University, DeLand, FL 32723
efriedma@stetson.edu

Abstract

We consider collections of convex, non-overlapping n-gons in the plane with the property that any point that is a vertex of one of the n-gons is the vertex of exactly k of them.

Triangles

We call a finite, non-empty, non-overlapping collection of triangles in the plane k-contained if any point that is a vertex of one of the triangles is the vertex of exactly k triangles. Thus any non-overlapping collection of triangles which do not share vertices is 1-contained. The smallest known k-contained sets of triangles for $2 \leq k \leq 5$ are shown in Figures 1 and 2.

Are these the smallest k-contained collections possible? Certainly for $k = 2$ and $k = 3$ they are, as can be easily verified by checking small cases. We suspect they are optimal for $k = 4$ and $k = 5$ as well, but do not have a proof.
There do not exist k-contained collections of triangles with k larger than 5. If there were, then the set of vertices and edges of the triangles would form a planar graph with every vertex having degree at least 6, which is impossible from Euler's Formula.

The triangles in the 2-contained set in Figure 1 are congruent. The smallest known 3-contained set of congruent triangles is shown in Figure 3. It is not known whether there is a 4-contained set of congruent triangles.
Other Polygons

In the same spirit, we can define a set of arbitrary convex polygons to be **k-contained** if any point that is a vertex of one of them is the vertex of exactly k of them. The smallest 2-contained and smallest known 3-contained sets of quadrilaterals are shown in Figure 4. The quadrilaterals in the 2-contained set in Figure 4 are congruent. It is unknown whether there is a 3-contained set of congruent quadrilaterals.

![Figure 4. 2-contained (left) and 3-contained (right) quadrilaterals](image)

The smallest 2-contained and 3-contained sets of pentagons are shown in Figure 5. Euler's Formula implies there are no 4-contained sets of quadrilaterals or pentagons.

![Figure 5. 2-contained (left) and 3-contained (right) pentagons](image)

Euler's Formula also implies the only other k-contained sets of n-gons are when k
The planar realizations of a truncated tetrahedron, truncated cube, and truncated dodecahedron give 2-contained sets of \( n \)-gons for \( n = 6, 8, \) and 10 respectively. To get 2-contained sets of 7-gons or 9-gons, we start with 2-contained sets of 8-gons or 10-gons respectively, and contract to points edges that connect pairs of polygons. It is not known whether these give the smallest possible 2-contained sets.

**Open Questions**

1. Are the \( k \)-contained sets of polygons presented here the smallest possible?
2. Does there exist a 4-contained set of congruent triangles, a 3-contained set of congruent quadrilaterals, or a 2-contained set of congruent pentagons?
3. How do these results change if we do not require the polygons to be convex?
4. What are the corresponding results in 3 dimensions?