Hinged Dissection of Polyominoes and Polyforms

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Abstract

A hinged dissection of a set of polygons S is a collection of polygonal pieces hinged together at vertices that can be rotated into any member of S. We present a hinged dissection of all edge-to-edge gluings of n congruent copies of a polygon P that join corresponding edges of P. This construction uses kn pieces, where k is the number of vertices of P. When P is a regular polygon, we show how to reduce the number of pieces to $\lceil k/2 \rceil (n-1)$. In particular, we consider polyominoes (made up of unit squares), polyiamonds (made up of equilateral triangles), and polyhexes (made up of regular hexagons). We also give a hinged dissection of all polyabolos (made up of right isosceles triangles), which do not fall under the general result mentioned above. Finally, we show that if P can be hinged into Q, then any edge-to-edge gluing of n congruent copies of P can be hinged into any edge-to-edge gluing of n congruent copies of Q.

1 Introduction

A *geometric dissection* [8, 13] is a cutting of a polygon into pieces that can be re-arranged to form another polygon. It is well known, for example, that any polygon can be dissected into any other polygon with the same area [2, 8, 14], but the bound on the number of pieces is quite weak. The main problem, then, is to find a dissection with the fewest possible number of pieces. Dissections have begun to be studied more formally than in their recreational past. For example, Kranakis, Krizanc, and Urrutia [12] study the asymptotic number of pieces required to dissect a regular m-gon into a regular n-gon. Czyzowicz, Kranakis, and Urrutia [4] consider the number of pieces to dissect a rational rectangle into a square using "glass cuts." An earlier paper by Cohn [3] studies the number of pieces to dissect a given triangle into a square.

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An intriguing subclass of dissections are *hinged dissections* [9, 8]. Instead of allowing the pieces to be re-arranged arbitrarily, suppose that the pieces are hinged together at their vertices, and we require pieces to remain attached at these hinges as they are re-arranged. Figure 1 shows the classic hinged dissection of an equilateral triangle into a square. This dissection is described by Dudeney [6], but may have been discovered by C. W. McElroy; see [9], [8, p. 136].

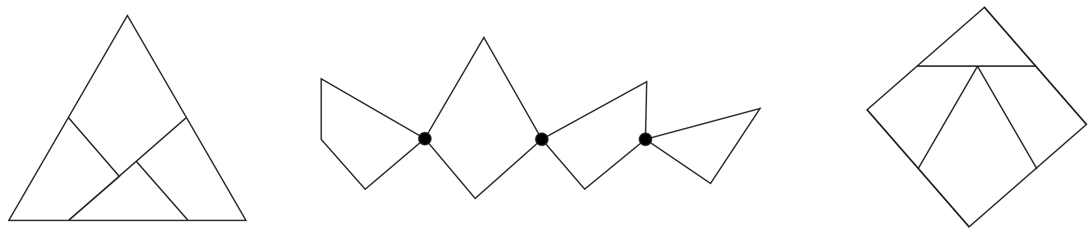

Figure 1: Hinged dissection of an equilateral triangle into a square.

For our purposes, we allow the pieces in a hinged dissection to overlap as the hinges rotate, but are interested in final configurations at which pieces do not overlap. We do not allow multiple hinges at a common vertex to cross each other, nor for hinges to "twist" and flip pieces over; see Figure 2.[1] Figures 1 and 2 illustrate our two drawing styles for hinged dissections: "geometrically exact" with dots for hinges and unshaded pieces, and "exaggerated" with segments for hinges and shaded pieces.

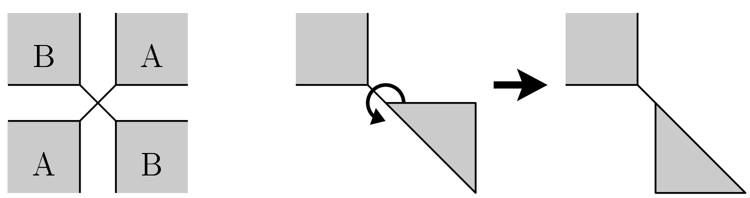

Figure 2: Forbidden features of hinged dissections: no crossing hinges (left) and no hinge twisting (right).

A natural question about hinged dissections is the following: can any polygon be hinge-dissected into any other polygon with the same area? This question is open and seems quite difficult. The main impediment to applying the same techniques as normal dissection is that hinged dissections are not obviously transitive: if A can be hinge-dissected to B, and B can be hinge-dissected to C, then it is not clear how to combine the two dissections into one from A to C. Of course, this transitivity property holds for normal dissections.

The possibility of an affirmative answer to this question is supported by the many examples of hinged dissections that have been discovered. Frederickson [9] has developed several techniques for constructing hinged dissections, and has applied them to design hundreds of examples. Akiyama and Nakamura [1] have demonstrated some hinged dissections under a restrictive model of hinging, designed to match the dissection in Figure 1. For example, they

[1] Frederickson [9] distinguishes different types of hinged dissections; this type is called *swing-hinged* (no twisting) and *wobbly hinged* (allow overlap during rotation).

show that it is possible to hinge-dissect any convex quadrangle into some parallelogram; in general, their work only deals with polygons having a constant number of vertices. Eppstein [7] gives a general method for hinge-dissecting any n-vertex polygon into its mirror image using $O(n)$ pieces. His method also reduces the general hinged-dissection problem to determining whether there is a hinged dissection between every pair of equal-area triangles satisfying a few simple extra properties.

In this paper, we explore hinged dissections of a class of polygons formed by gluing together several nonoverlapping equal-size regular k-gons along touching pairs of edges, for a fixed k. We call such a polygon a *poly-k-regular* or *polyregular* for short. The polygon need not be simply connected; we allow it to have holes. An *$n \times k$-regular* is a poly-k-regular made of n regular k-gons. Poly-k-regulars include the well-studied *polyominoes* ($k = 4$), *polyiamonds* ($k = 3$), and *polyhexes* ($k = 6$) [10, 11, 15]. Polyominoes are of particular interest to computational geometers, because they include orthogonal polygons whose vertices have rational coordinates.

This paper proves that not only can any $n \times k$-regular be hinge-dissected into any other $n \times k$-regular, but furthermore that there is a single hinged dissection that can be rotated into all $n \times k$-regulars, for fixed n and k. This includes both reflected copies of each polyregular. Section 4 describes two methods for solving this problem, the more efficient of which uses $\lceil k/2 \rceil (n-1)$ pieces. The more-efficient method combines a simpler method, which uses $k(n-1)$ pieces, and an efficient method for the special case of polyominoes, described in Section 3 where we also give some lower bounds.

Next, in Section 5 we consider another kind of "polyform." A *polyabolo* is a connected edge-to-edge gluing of nonoverlapping equal-size right isosceles triangles. In particular, every n-omino is a $2n$-abolo, as well as a $4n$-abolo. We prove that there is a $4n$-piece hinged dissection that can be rotated into any n-abolo for fixed n.

In Section 6, we show an analogous result for a general kind of polyform, which allows us to take certain edge-to-edge gluings of copies of a general polygon. This result is a generalization of polyregulars, although it uses more pieces. It does not, however, include the polyabolo result, because of some restrictions placed on how the copies of the polygon can be joined.

Section 7 shows there are hinged dissections that can rotate into polyforms made up of different kinds of polygons. Specifically, it finds efficient hinged dissections that rotate into (a) all polyiamonds and all polyominoes, (b) all polyominoes and all polyhexes, and (c) all polyiamonds and all polyhexes. On the way, we show that any dissection in which the pieces are hinged together according to some graph, such as a path, can be turned into a dissection in which the pieces are hinged together in a cycle.

A preliminary version of this work appeared in [5].

2 Basic Structure of Polyforms

We begin with some definitions and basic results about the structure of polyforms. As we understand it, the term "polyform" is not normally used in a formal sense, but rather as a figurative term for objects like polyominoes, polyiamonds, polyhexes, and polyabolos.

However, in this paper, we find it useful to use a common term to specify all of these objects collectively, and "polyform" seems a natural term for this purpose.

Specifically, we define a (planar) *polyform* to be a finite collection of copies of a common polygon P such that the interior of their union is connected, and the intersection of two copies is either empty, a common vertex, or a common edge. An n-form is a polyform made of n copies of P. We call P the *type* of the polyform. Polyforms are considered equivalent modulo rigid motions (translations and rotations), but not reflections.

The *graph* of a polyform is defined as follows. Create a vertex for each polygon in the collection, and connect two vertices precisely if the corresponding polygons share an edge. Because every connected graph has a vertex whose removal leaves the graph connected, we have the following immediate consequence.

Proposition 1 *Every n-form has a polygon whose removal results in a (connected) $(n-1)$-form of the same type.*

This simple result is useful for performing induction on the number of polygons in a polyform. More precisely, if we view the decomposition in the reverse direction (adding polygons instead of removing them), then this lemma says that any polyform can be built up by a sequence of additions such that any intermediate form is also connected. To construct a hinged dissection, we will repeatedly hinge a new polygon onto the previously constructed polyform.

In the next lemma, the first sentence restates Proposition 1 in the context of adding polygons instead of removing them. In addition, the second sentence provides some additional constraints on the addition process, which will be important for optimizing some of our dissections.

Lemma 2 *Any polyform of type P can be built up by a sequence of* gluings *in which a new copy of P is placed against an edge of an already placed copy of P called the* parent*. (As a special case, the first copy of P can be placed arbitrarily.) Furthermore, this gluing sequence can be chosen so that only one copy of P is glued with the first copy of P as the parent.*

Proof: Pick any spanning tree T of the graph of an n-form. Let P_1 be some leaf of T. Let P_2 denote the unique vertex incident to P_1 in T, and glue it to P_1. Now perform a depth-first traversal of T rooted at P_2, and label newly visited vertices as $P_3, P_4, \ldots, P_n$, each gluing to its parent. The result is a gluing sequence for the polyform, such that only one polygon (P_2) glues to P_1. $\square$

Our constructions and proofs of correctness for hinged dissections of n-forms will follow a common outline. The construction is simple: we describe a (often cyclicly) hinged dissection parameterized by n; more precisely we define a function from the positive integers to hinged dissections, call it $H(n)$. Now we need to prove that, for any $n \geq 1$, $H(n)$ can be rotated into all n-forms. This will be done using the ideas of Lemma 2. First we show how to construct a single polygon (P_1); that is, we show that $H(1)$ can be rotated into P. Second we show how to add each polygon P_n onto a rotation of $H(n-1)$ into an arbitrary $(n-1)$-form F_{n-1}, so that in the end we have a rotation of $H(n)$ into a desired n-form. The key is that no matter where we attach P_n to F_{n-1}, we obtain a rotation of exactly the same hinged dissection,

$H(n)$. This means that the same $H(n)$ can be rotated into all of these configurations—all n-forms.

This technique will be used repeatedly, so the following lemma specifies it formally. It also generalizes to allow starting with c-forms for a constant $c \geq 1$ instead of just a single copy of P. We will often use $c = 2$ to optimize some of our dissections, although we will never use higher values of c.

Lemma 3 *For any constant $c \geq 1$, a parameterized hinged dissection $H(n)$ rotates into all n-forms of type P, for every $n \geq c$, if*

1. *$H(c)$ rotates into any c-form of type P, and*
2. *for every $n > c$, given any rotation of $H(n-1)$ into an $(n-1)$-form F, and given any edge e of F, $H(1)$ can be rotated into P, placed next to e, and hinges can be added and removed between touching vertices (that is, the two hinged dissections can be split up and spliced back together) such that the resulting hinged dissection is $H(n)$. This process is called* attaching *a copy of P to F.*

Furthermore, if $c \geq 2$, some copy of P in the initial c-form (from Condition 1) will never have a copy of P attached to it, and in this sense is called slippery.

Proof: Consider any n-form F of type P for $n \geq c$. We will prove that $H(n)$ rotates into F, and hence $H(n)$ rotates into all n-forms of type P. Consider a gluing sequence for F from Lemma 2. The union of the first c copies of P is some c-form C of type P. By Condition 1, $H(c)$ can be rotated into C. Now attach the remaining copies of P one by one in the order specified by the gluing sequence. At each step, if we have a rotation of $H(k)$ into the first k copies of P, then attaching the next gives us a rotation of $H(k+1)$. By induction, we reach a rotation of $H(n)$ into the desired n-form F.

Now C contains the first copy of P in the gluing sequence, call it P_1. The gluing sequence from Lemma 2 glues only one copy of P to P_1. Provided $c \geq 2$, C has already glued a copy of P to P_1, and hence no other copies of P are glued to P_1. The above construction makes precisely one attachment corresponding to each gluing other than the first $c-1$ gluings; thus, no copies of P are ever attached to P_1. $\square$

3 Polyominoes

Let us start with the special case of polyominoes. This serves as a nice introduction to efficient hinged dissection of polyregulars, and is also where our research began.

3.1 Small Polyominoes

Constructing a hinged dissection that forms any n-omino is easy for small values of n. There is only one monomino and one domino, so no hinges are necessary. There are two trominoes, and a two-piece dissection is easy to find; see Figure 3. The natural four-square dissection can be hinged to rotate into all tetrominoes; see Figure 4.

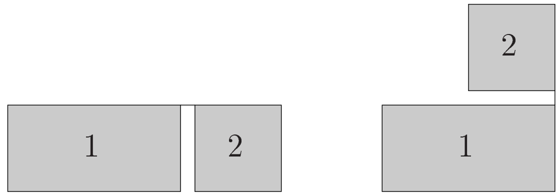

Figure 3: Two-piece hinged dissection of all trominoes.

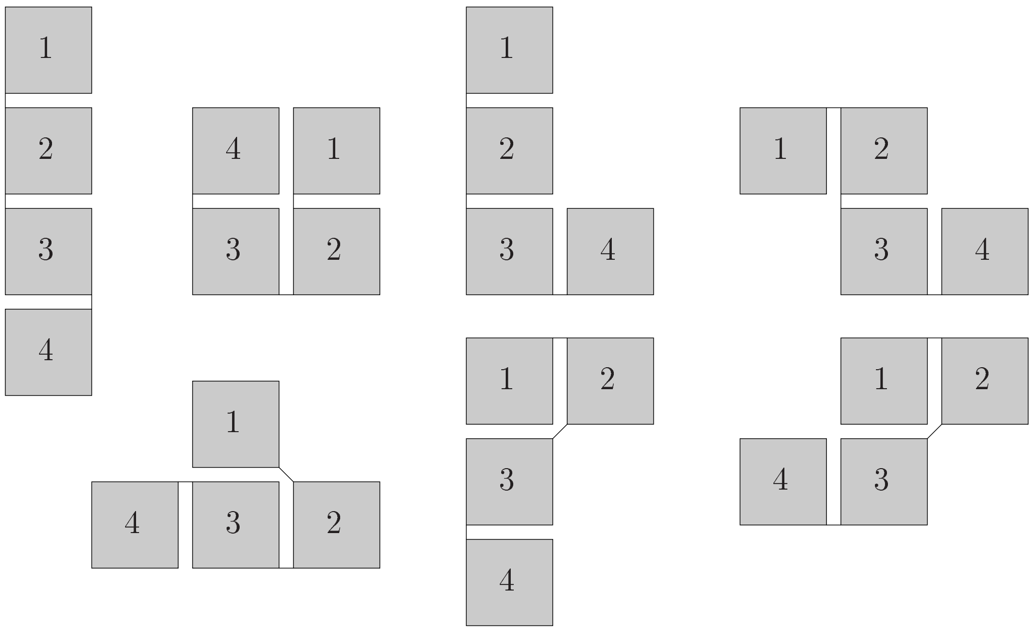

Figure 4: Four-piece hinged dissection rotated into all tetrominoes, while keeping the orientation of square 2 fixed.

Unfortunately, in contrast to normal dissections, dividing an n-omino into its constituent squares is insufficient for it to hinge into all other n-ominoes for $n = 5$:

Theorem 4 *Five identical squares cannot be hinged in such a way that they can be rotated into all pentominoes.*

Proof: Suppose there were a hinging H of five squares that could rotate into every pentomino. We first consider the I-pentomino, which is a 1×5 rectangle. Because H rotates into the I-pentomino, the squares must be hinged one after the other in a chain, ordered by their position in the rectangle. An example without the hinges is given in Figure 5.

We next consider the X-pentomino, in which the five squares form a (Greek) cross. Four of the five squares are arms of the X-pentomino. Each of these four has two adjacent vertices that do not touch any other square, and thus cannot have hinges on them. It follows that none of these four squares can have hinges at diagonally opposite vertices. Thus, at most one of the five squares can have hinges at diagonally opposite vertices. Figure 6 gives an example of such a hinging of the X-pentomino (there are several). In this example, only square 3, the middle square, has hinges at diagonally opposite vertices.

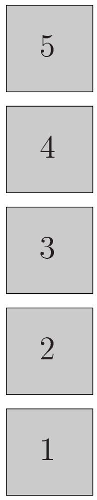

Figure 5: I-pentomino without hinges

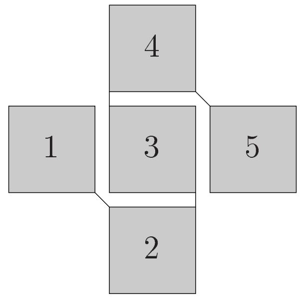

Figure 6: X-pentomino with sample hinging

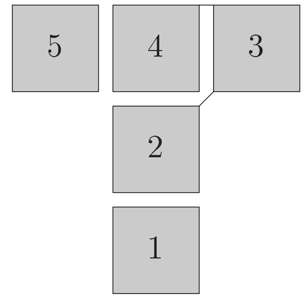

Figure 7: T-pentomino with forced hinges

We next consider the T-pentomino, in which three squares are stacked one on top of the other, and the other two squares are to the right and to the left of the top square in the stack. One end of the chain (call it square 1) must be on the bottom of the stack, because it is adjacent to only one other square (which must necessarily be square 2). The top middle square cannot be square 3, for otherwise it would be impossible to connect all the squares in a chain. Thus, in particular, the other end of the chain (square 5) must be either the left or the right square at the top. This argument limits us to the configuration in Figure 7 and its mirror image.

Suppose that we transform H from the T-pentomino to the I-pentomino while leaving square 2 in the same orientation. If the I-pentomino lies horizontally, then square 1 must rotate 180° clockwise, causing square 2 to have hinges at diagonally opposite vertices. Square 3 must rotate 90° clockwise, square 4 rotates 270° clockwise, and square 5 rotates 90° clockwise, as shown in Figure 8. But this requires that two squares, squares 2 and 4, have hinges at diagonally opposite vertices. Because this possibility has been ruled out, we cannot transform H from the T-pentomino to the I-pentomino while leaving square 2 in the same orientation and having the I-pentomino lie horizontally.

Suppose that we transform H from the T-pentomino to the I-pentomino while leaving square 2 in the same orientation, with the I-pentomino standing vertically. Then square 3 must rotate 90° counterclockwise. This would leave both of its hinges adjacent to square 2, as shown in Figure 9. Clearly squares 3 and 4 cannot be connected in this way.

This exhausts all cases for transforming H from the T-pentomino to the I-pentomino. Thus the desired hinging H does not exist. □

3.2 General Polyominoes

Our first hinged dissection of general n-ominoes uses $2n$ right isosceles triangles; see Figure 10. Note that a *cyclicly hinged* dissection (in which the pieces are connected in a cycle)

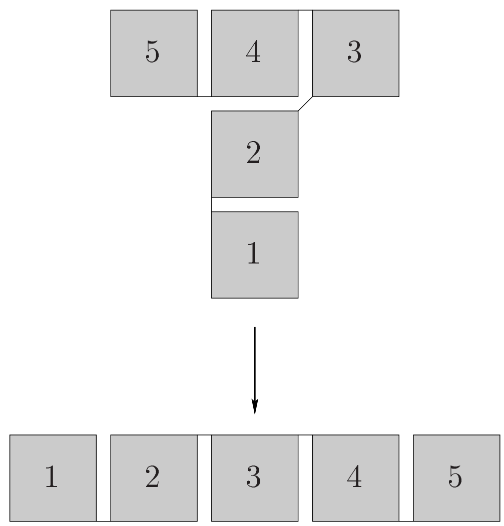

Figure 8: First try: T to I

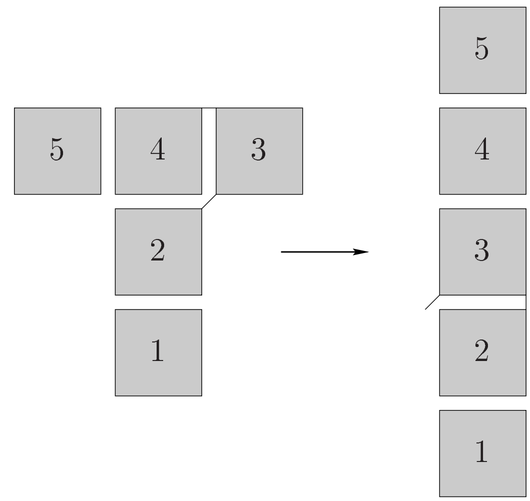

Figure 9: Second try: T to I

is a stronger result than a *linearly hinged* dissection (in which the pieces are connected in a path): simply breaking one of the hinges in the cycle results in a linearly hinged dissection.

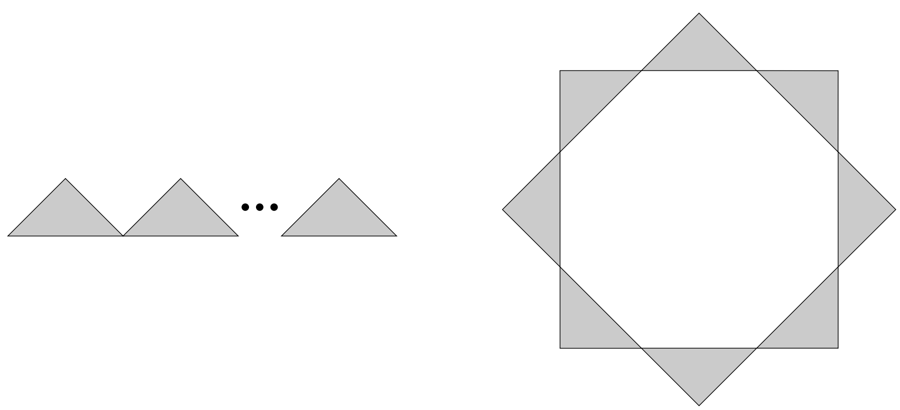

Figure 10: $2n$-piece hinged dissection of all n-ominoes. (Left) Connected in a path. (Right) Connected in a cycle, $n = 4$.

Theorem 5 *A cycle of* $2n$ *right isosceles triangles, joined at their base vertices, can be rotated into any* n*-omino.*

Proof: Apply Lemma 3 with $c = 1$. The case $n = 1$ is shown in Figure 11.

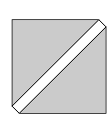

Figure 11: A 2-piece hinged dissection of a monomino.

We can attach a square S to the hinging of a polyomino P as follows; refer to Figure 12. Let T be the triangle in this hinging that shares an edge with S. One of its base vertices, say v, is also incident to S, and it must be hinged to some other triangle T'. We split S into two right isosceles triangles S_1 and S_2 so that both have a base vertex at v. Now we replace T's hinge at v with a hinge to S_1, and add a hinge from S_2 to T' at v. Finally, S_1 and S_2 are hinged together at their other base vertex. The result is a hinging of $2n$ triangles rotated into P. We can optionally swap S_1 and S_2 in order to avoid crossings between the hinges. □

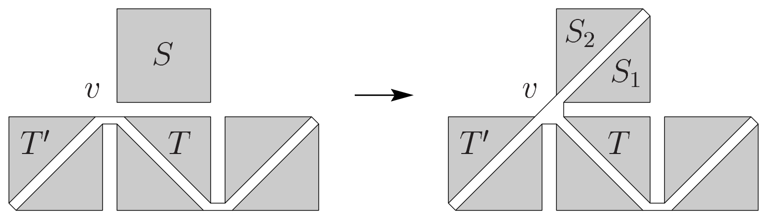

Figure 12: Attaching a square S to the $2n$-piece hinged dissection of n-ominoes.

Now we explain how to modify this dissection to use two fewer pieces:

Corollary 6 *For $n \geq 2$, the $(2n-2)$-piece hinged dissection in Figure 13 can be rotated into any n-omino.*

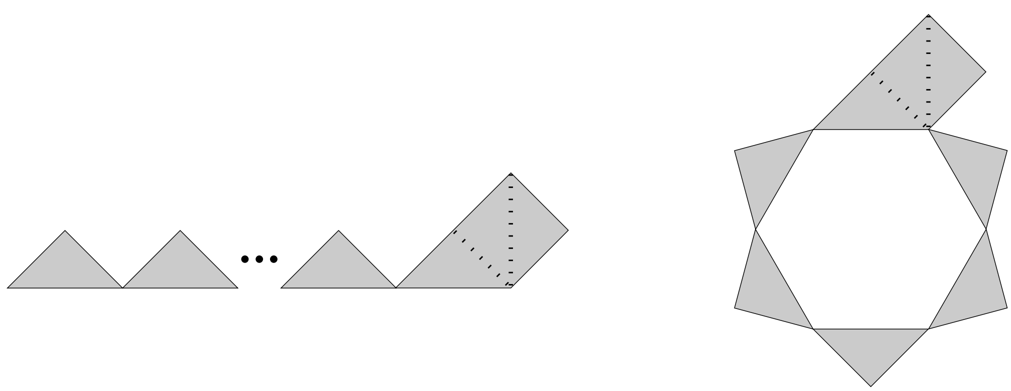

Figure 13: $(2n-2)$-piece hinged dissection of all n-ominoes. (Left) Linear. (Right) Cyclic, $n = 4$.

Proof: Apply Lemma 3 with $c = 2$. The case $n = 2$ is shown in Figure 14. One square in this domino, S_1, has hinges at diagonally opposite vertices just as before, but the other square, S_2, has only one hinge. By symmetry, we can arrange in Lemma 3 for S_2 to be chosen as slippery, and hence all attachments act as in Theorem 5. □

This method will be generalized in Section 4.2 to support all polyregulars.

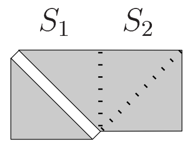

Figure 14: A 2-piece hinged dissection of a domino.

4 Polyregulars

This section describes two methods for constructing, given any $n \geq 1$ and $k \geq 3$, a hinged dissection that can be rotated into all $n \times k$-regulars. In particular, this result generalizes the polyomino case ($k = 4$) of the previous section. However, our first solution will not be as efficient as that in the previous section; it serves as a simpler warm-up for the second solution. The second solution combines the first solution and the efficient $k = 4$ solution to obtain a single method that is efficient for all k.

4.1 Inefficient Polyregulars

Our first hinged dissection splits each regular k-gon into k isosceles triangles, by adding an edge from each vertex to the center of the polygon. See Figure 15. Because it is rather difficult to draw a generic regular k-gon, our figures will concentrate on the case of $k = 3$, i.e., n-iamonds. We define the *base* of each isosceles triangle to be the edge that coincides with an edge of the regular k-gon (whose length differs from all the others unless $k = 6$). The *base vertices* are the endpoints of the base, and the *opposite angle* is the interior angle of the remaining vertex.

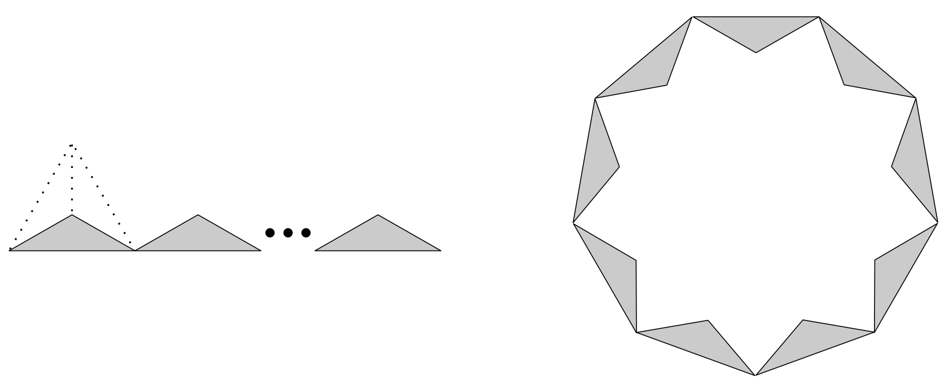

Figure 15: $3n$-piece hinged dissection of all n-iamonds. (Left) Linear. (Right) Cyclic, $n = 3$.

Theorem 7 *A cycle of kn isosceles triangles with opposite angle $2\pi/k$, joined at their base vertices, can be rotated into any $n \times k$-regular.*

Proof: Apply Lemma 3 with $c = 1$. The case $n = 1$ is shown in Figure 16.

We can attach a regular k-gon R to the hinging of an $n \times k$-regular P as follows; refer to Figure 17. Let T be the triangle in this hinging that shares an edge with R. Both of its

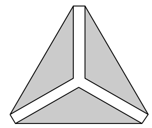

Figure 16: A 3-piece hinged dissection of a moniamond.

base vertices are also incident to R. Let v be either one of T's base vertices, and suppose that it is hinged to triangle T'. We split R into k isosceles triangles $R_1, \ldots, R_k$ so that both R_1 and R_k have a base vertex at v. Now we replace T's hinge at v with a hinge to R_1, and add a hinge from R_k to T' at v. Finally, R_i and R_{i+1} are hinged together at their common base vertex, for all $1 \leq i < k$. The result is a hinging of kn triangles rotated into P. We can optionally renumber $R_1, \ldots, R_k$ as $R_k, \ldots, R_1$ in order to avoid crossings between the hinges. $\square$

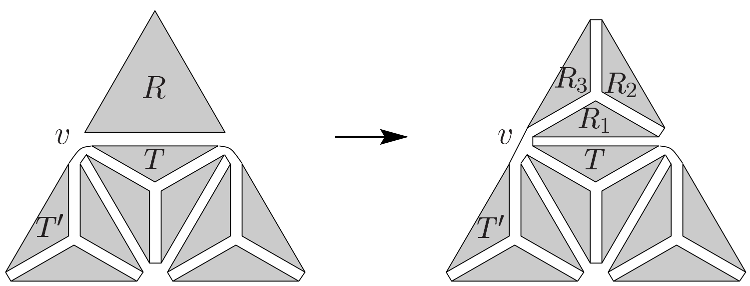

Figure 17: Attaching an equilateral triangle R to the $3n$-piece hinged dissection of n-iamonds.

While the number of pieces will be improved dramatically in the next section, we show that the trick of merging the last few pieces also applies to this dissection, reducing the number of pieces by k:

Corollary 8 *For $n \geq 2$, the $k(n-1)$-piece hinged dissection in Figure 18 can be rotated into any $n \times k$-regular.*

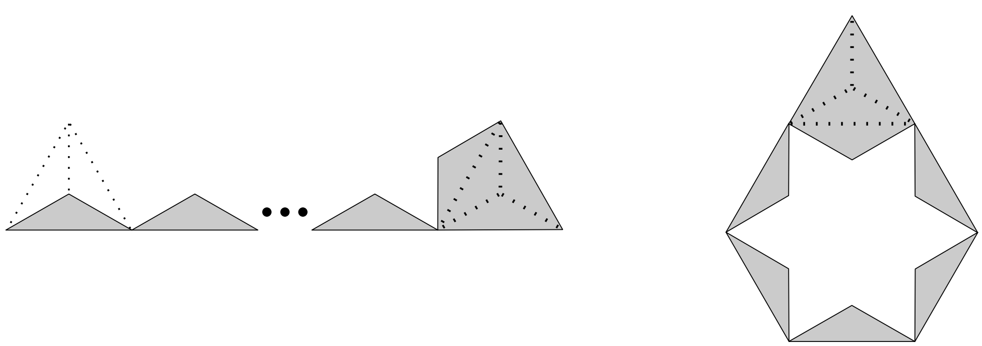

Figure 18: $(3n-3)$-piece hinged dissection of all n-iamonds. (Left) Linear. (Right) Cyclic, $n = 3$.

Proof: Apply Lemma 3 with $c = 2$. The case $n = 2$ is shown in Figure 19. One regular k-gon R_1 in this $2 \times k$-regular has hinges at all its vertices just as before, but the other regular k-gon R_2 has only two hinges. By symmetry, we can arrange in Lemma 3 for R_2 to be chosen as slippery, and hence all attachments act as in Theorem 7. $\square$

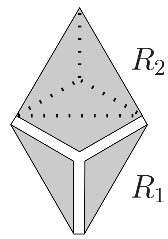

Figure 19: A 3-piece hinged dissection of a diamond.

4.2 Improved Polyregulars

The goal of this section is to improve the previous hinged dissection for polyregulars so that, for $k = 4$, the number of pieces matches the method in Section 3 for polyominoes. To see how to do this, let us compare the two methods when restricted to $k = 4$. The method in Section 4.1 splits each square into four right isosceles triangles; i.e., it makes four cuts to the center of the square. In contrast, the method in Section 3 makes only two of these cuts. In other words, the method in Section 3 can be thought of as merging adjacent pairs of right isosceles triangles from the method in Section 4.1.

This discussion suggests the following generalized improvement to the method in Section 4.1, for arbitrary k: join adjacent pairs of right isosceles triangles, until zero or one triangles are left. For even k (like $k = 4$), this will halve the number of pieces; and for odd k, it will almost halve the number of pieces. The intuition behind why this method will work is that when we added a regular k-gon to an existing polyregular in the proof of Theorem 7, we had two existing hinges at which we could connect the new k-gon; at most halving the number of hinges will still leave at least one hinge to connect the new k-gon.

In general, our hinged dissection will consist of $\lceil k/2 \rceil n$ pieces. If k is even, every piece will be the union of two isosceles triangles, each with opposite angle $2\pi/k$, joined along an edge other than the base. If k is odd, every group of $\lfloor k/2 \rfloor$ of these pieces is followed by a single isosceles triangle with opposite angle $2\pi/k$. For example, for polyiamonds ($k = 3$), the pieces alternate between single triangles and "double" triangles (see Figure 20). Independent of the parity of k, the pieces are joined at the base vertices of the constituent triangles that have not been merged to other base vertices.

Again, our figures will focus on the case $k = 3$, as in Figure 20.

Theorem 9 *The described cyclicly hinged dissection of $\lceil k/2 \rceil n$ pieces can be rotated into any $n \times k$-regular.*

Proof: Apply Lemma 3 with $c = 1$. The case $n = 1$ is shown in Figure 21.

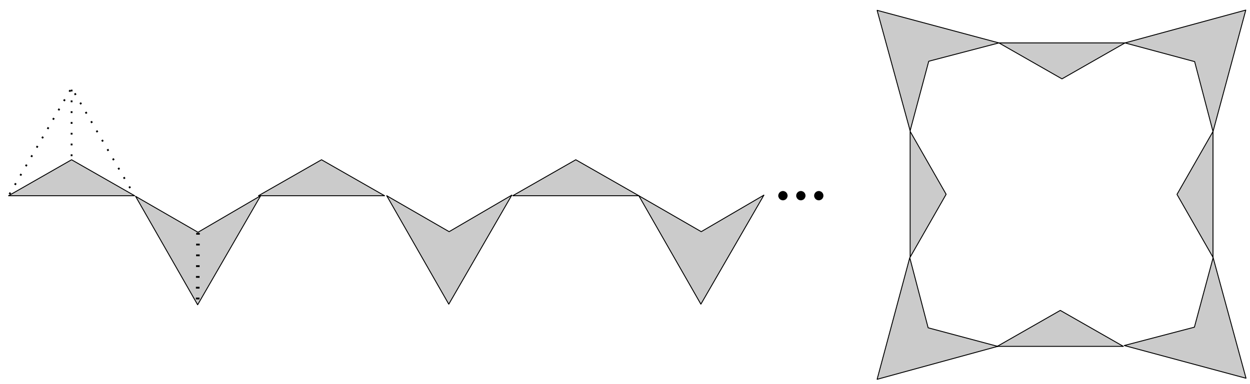

Figure 20: $2n$-piece hinged dissection of all n-iamonds. (Left) Linear. (Right) Cyclic, $n = 4$.

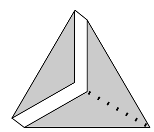

Figure 21: A 2-piece hinged dissection of a moniamond.

We can attach a regular k-gon R to the hinging of an $n \times k$-regular P similar to the proof of Theorem 7; refer to Figure 22. Let T be the piece in this hinging that shares an edge with R. Both of the base vertices of one of its constituent triangles are incident to R. Let v be such a base vertex of T that is not joined to a base vertex of another constituent triangle of T (which we call *lone* base vertices), and suppose that it is hinged to piece T'. We split R into $\lceil k/2 \rceil$ pieces $R_1, \ldots, R_{\lceil k/2 \rceil}$ so that both R_1 and $R_{\lceil k/2 \rceil}$ have a lone base vertex at v. Now we replace T's hinge at v with a hinge to R_1, and add a hinge from $R_{\lceil k/2 \rceil}$ to T' at v. We hinge R_i and R_{i+1} together at their common lone base vertex, for all $1 \le i < \lceil k/2 \rceil$. Finally, if k is odd, we choose one of the pieces $R_1, \ldots, R_{\lceil k/2 \rceil}$ to be a single triangle instead of a double triangle, appropriately so that the single triangles appear periodically in the resulting cycle of pieces, with a period of $\lceil k/2 \rceil$. The result is the desired hinging of $\lceil k/2 \rceil n$ pieces rotated into P. We can optionally renumber $R_1, \ldots, R_{\lceil k/2 \rceil}$ as $R_{\lceil k/2 \rceil}, \ldots, R_1$ in order to avoid crossings between the hinges. □

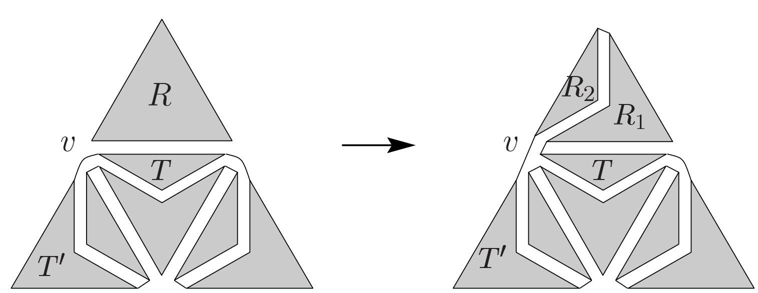

Figure 22: Attaching an equilateral triangle R to the $2n$-piece hinged dissection of n-iamonds.

Our final hinged dissection of polyregulars improves the previous one by $\lceil k/2 \rceil$ pieces.

Corollary 10 *For $n \geq 2$, the $\lceil k/2 \rceil (n-1)$-piece hinged dissection in Figure 23 can be rotated into any $n \times k$-regular.*

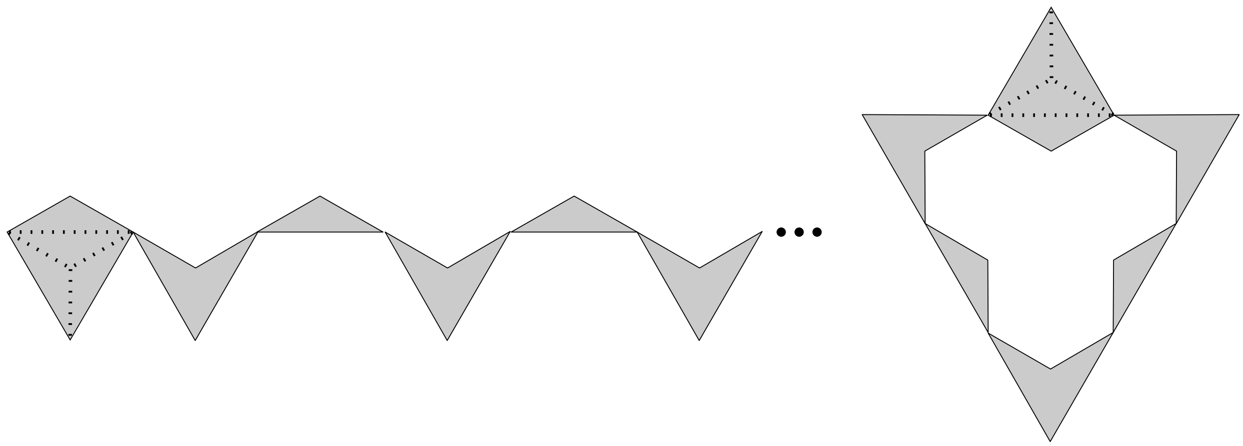

Figure 23: $(2n-2)$-piece hinged dissection of all n-iamonds. (Left) Linear. (Right) Cyclic, $n = 4$.

Proof: Apply Lemma 3 with $c = 2$. The case $n = 2$ is shown in Figure 24. One regular k-gon R_1 in this $2 \times k$-regular has hinges at roughly half of its vertices just as before, but the other regular k-gon R_2 has only two hinges. By symmetry, we can arrange in Lemma 3 for R_2 to be chosen as slippery, and hence all attachments act as in Theorem 9. □

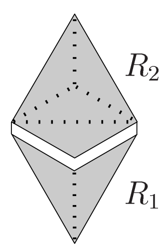

Figure 24: A 2-piece hinged dissection of a diamond.

5 Polyabolos

Another well-studied class of polyforms that does not fall under the class of polyregulars is *polyabolos*, the union of equal-size half-squares (right isosceles triangles) joined at equal-length edges. In this section, we present a hinged dissection of polyabolos.

Our dissection is a cycle of $4n$ right isosceles triangles, as shown in Figure 25. Like Figure 10, the triangles point outward, but unlike Figure 10, they are joined at a short edge instead of the long edge. The orientations of the triangles (or equivalently, which of the two short edges we connect to the other triangles) alternate along the cycle.

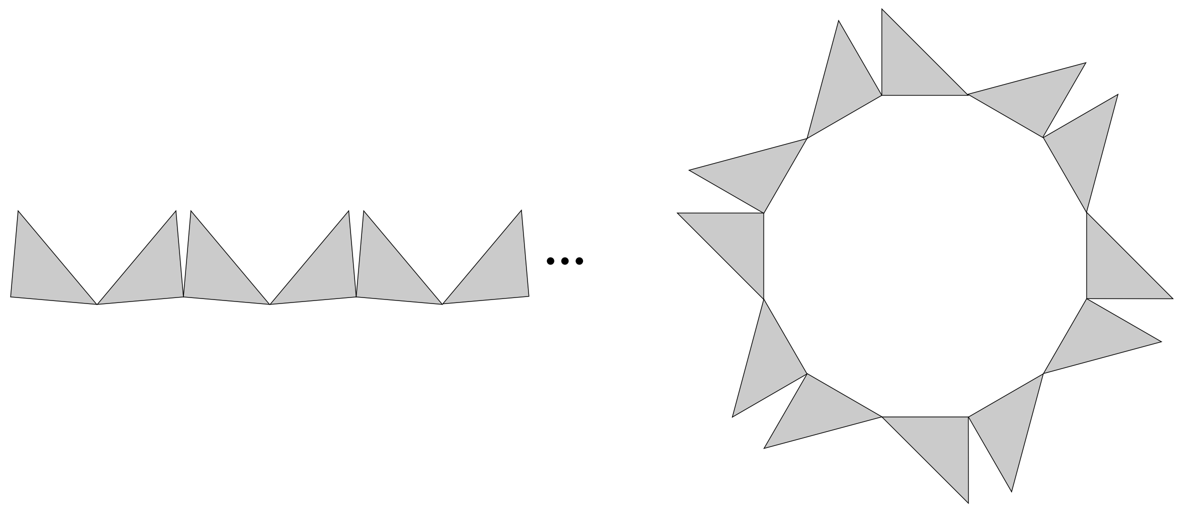

Figure 25: $4n$-piece hinged dissection of all n-abolos. (Left) Linear. (Right) Cyclic, $n = 3$.

Theorem 11 *The $4n$-piece hinged dissection in Figure 25 can be rotated into any n-abolo.*

Proof: Apply Lemma 3 with $c = 1$. The case $n = 1$ is shown in Figure 26. Note that in contrast to all previous dissections, there are no hinges at the vertices of the monabolo. There are, however, hinges at the midpoints of all the edges.

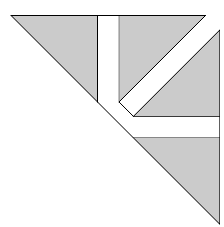

Figure 26: A 4-piece hinged dissection of a monabolo.

We can attach a half-square H to the hinging of a polyabolo P at these midpoints as shown in Figure 27. There are three cases according to relative orientations of H and the incident half-square. But in all cases we obtain the same hinged dissection (Figure 25) with triangles pointing outward from the cycle, and alternating in orientation along the cycle. $\square$

An interesting consequence of this theorem is a hinged dissection that can be rotated into polyominoes with squares of different sizes:

Corollary 12 *A common hinged dissection can be rotated into any n-omino and any $2n$-omino.*

Proof: Both can be viewed as a $4n$-abolo, by splitting a square in the n-omino into four pieces (Figure 28, left) and splitting a square in the $2n$-omino into two pieces (Figure 28, right). $\square$

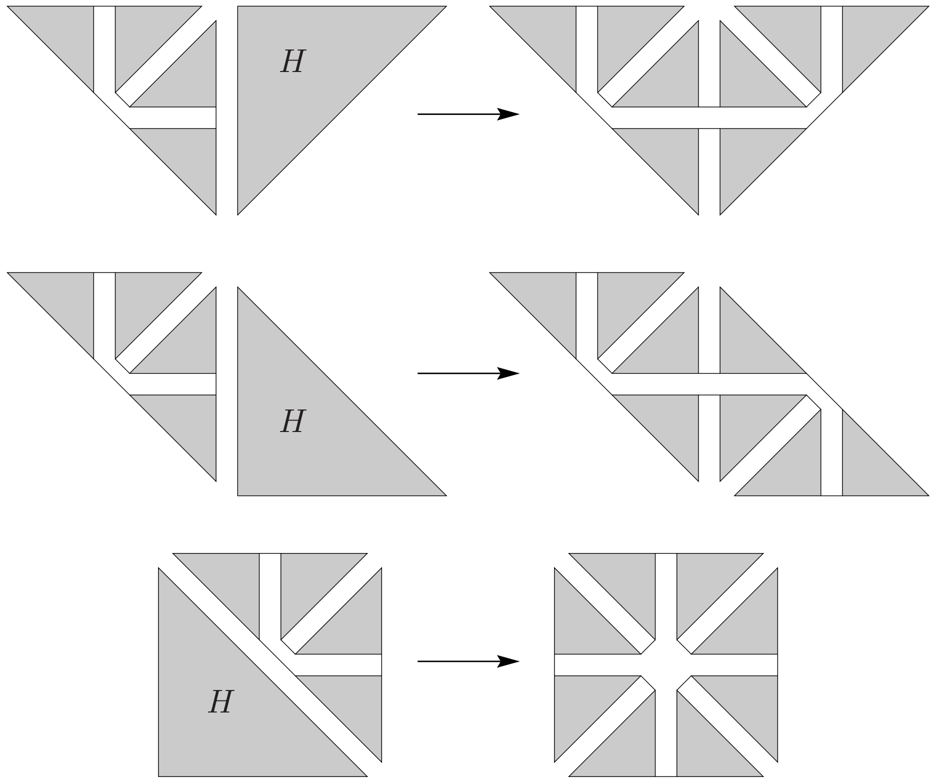

Figure 27: Attaching a half-square H to the $4n$-piece hinged dissection of n-abolos.

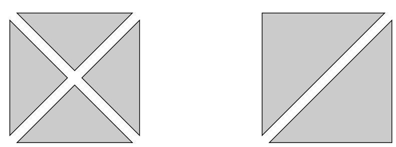

Figure 28: Two ways to split a polyomino into a polyabolo.

This dissection uses a large number of pieces, namely $16n$. In fact, we can do much better by simply using the $4n$-piece path dissection of the $2n$-omino from Figure 10, left. On the one hand, as in Theorem 5, the hinged dissection can rotate into any $2n$-omino. On the other hand, we can rotate the pieces and view adjacent pairs of pieces as (temporarily) merged along their short sides, and we obtain a hinged dissection that can rotate into any n-omino. The full cyclic hinging does not work, because the $2n$-omino wants the long sides inside while the n-omino wants them outside, so rotating from one to the other would require twisting the hinges.

6 Other Polyforms

An interesting open problem is whether there is a hinged dissection that can be rotated into all polyforms of a particular size and type. In other words, for a fixed n and polygon P, is there a hinged dissection that rotates into any connected edge-to-edge gluing of n copies of

P? As a step towards solving this problem, we present a hinged dissection for a large class of polyforms of type P. Specifically, we impose the restriction that for any two copies of P sharing an edge, there must be a rigid motion (a combination of translations and rotations) that

1. takes one copy of P to the other copy, and
2. takes the shared edge in one copy to the shared edge in the other copy.

Such a polyform is called a *restricted polyform*. The first constraint says that the copies of P are not flipped over. The second constraint says that only "corresponding edges" of copies of P are joined. This is actually not that uncommon: if P is generic in the sense that no two edges have the same length, then the second constraint is implied by the edge-to-edge condition.

Comparing to our previous results, every polyregular is a polyform satisfying the described restriction. However, polyabolos do not satisfy the second constraint; for example, if we join two right isosceles triangles so that their union forms a larger right isosceles triangle, then noncorresponding edges have been joined.

The method for hinge-dissecting restricted polyforms works as follows. We subdivide P by making cuts incident to the midpoint of every boundary edge, so that there is one piece surrounding each vertex of P. This can be done as follows; refer to Figure 29. Take a triangulation T of P. First, cut along edges of the triangulation so that the remaining connections between triangles form a dual tree D; this step adds artificial "edges" to P to make P simply connected (hole-free). Second, position each vertex of D anywhere interior to the corresponding triangle in T, and cut along the edges of D. Third, cut from each vertex of D to the midpoint of every edge of P incident to the corresponding triangle of T.

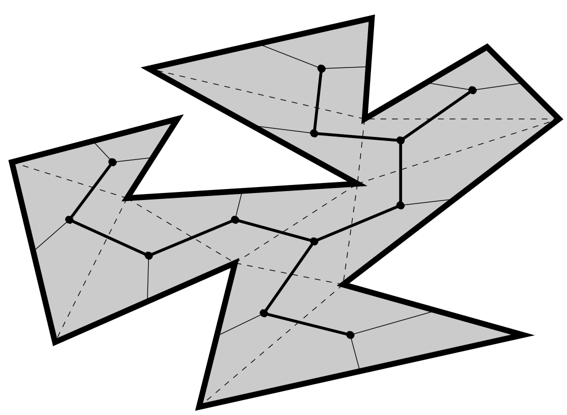

Figure 29: Cutting up a polygon P with cuts through the midpoint of every edge. Only the dashed lines are not cuts; they show the underlying triangulation T. The thick solid lines and dots form the dual tree D, and the thin solid lines are the cuts from dual vertices to edge midpoints.

Now the actual hinged dissection is simple: repeat the cyclic decomposition of P, n times, and hinge the pieces at the midpoints of the edges of P. Now at any edge of P we can decide to visit an incident copy of P before completing the traversal of P, and we visit the same sequence of pieces. See Figure 30 for a simple example.

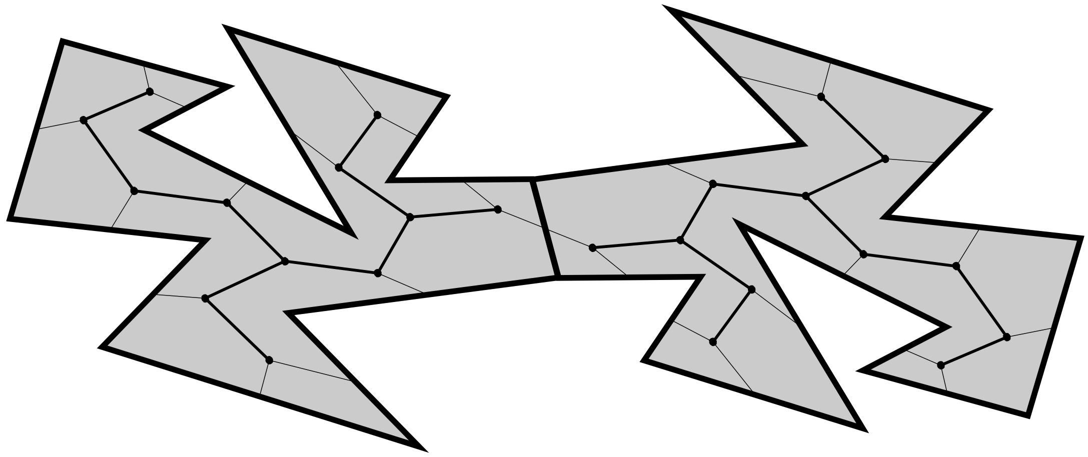

Figure 30: Joining two copies of P, once each is cut up: switching over from one copy of P to the other does not affect the order of shapes of pieces we visit.

In this way we can construct any restricted polyform of type P, nearly proving the following theorem:

Theorem 13 *There is a kn-piece cyclicly hinged dissection that can be rotated into any restricted n-form of type P, where P is a polygon with k vertices.*

Proof: There is one detail omitted in the discussion above, so let us go through a formal proof. Apply Lemma 3 with $c = 1$. (While this lemma was designed for general n-forms, it applies equally well to restricted n-forms.) The case $n = 1$ just takes the decomposition of P described above, and hinges it at the midpoints of edges of P. Adding a copy of P to a restricted polyform of type P requires special care. Let Q denote the copy of P to which we want to attach P, and let e denote the edge of Q to which P will attach. We need to place P against e such that the rigid motion mapping Q to P and e to e also maps the pieces of Q to the pieces of P (where "pieces" refer to the subdivision described above).

Certainly there is a rigid motion m mapping the pieces of Q to the pieces of P, so we should attach P's edge $m(e)$ to Q's edge e. The only possible wrinkle is that if P has symmetry, in the sense that there is a rigid motion s from P to itself, then we might instead attempt to attach $s(m(e))$ to e. Fortunately, we get to choose the orientation of the subdivision of P for the copy we are attaching. We can explore all symmetric versions of the subdivision of P and choose the one that places $m(e)$ against e. $\square$

7 Polyforms of Different Types

7.1 Different Restricted Polyforms

Figure 1 shows that a regular 3-gon can be hinge-dissected into a regular 4-gon. This example suggests the more general possibility of hinge-dissecting $n \times 3$-regulars into $n \times 4$-regulars.

Indeed, such a hinged dissection is possible, even for restricted polyforms of arbitrary types:

Theorem 14 *Given a hinged dissection H that rotates into polygons $P_1, P_2, \ldots, P_k$, there is a cyclicly hinged dissection H' that rotates into all restricted n-forms of type P_i for all $1 \leq i \leq k$. If H has n pieces and P_i has p_i sides, then H' has at most $3n - 3 + \sum_{i=1}^{k} p_i$ pieces.*

Before we discuss applications of this theorem, let us prove it. First we need a result which is interesting in its own right: any hinged dissection can be turned into a cyclicly hinged dissection. By removing one hinge from the cyclicly hinged dissection, we can also obtain a linearly hinged dissection.

Lemma 15 *For any dissection H, there is a cyclicly hinged dissection H' that can be rotated into any polygon that H can. If H has n pieces, then H' has at most $3n - 3$ pieces.*

Proof: The graph of the hinging structure (in which vertices represent pieces and edges represent hinges) can be any planar graph. First take a spanning tree of that graph, and remove all other hinges. This transformation removes the cycles from the hinging structure while keeping the pieces connected. Now for each (original) piece with only one hinge, we cut it along a polygonal line from that hinge to any other point on the boundary (e.g., another vertex), and add a hinge between the two pieces at that point. For each original piece that has at least two hinges, we cut along a tree of line segments that is interior to the piece and has leaves at the hinges, and we replace each original hinge with two "parallel" hinges. The result is a cyclicly hinged dissection H', which can be rotated as H can because it simply subdivides and adds more degrees of motion. See Figure 31 for an example.

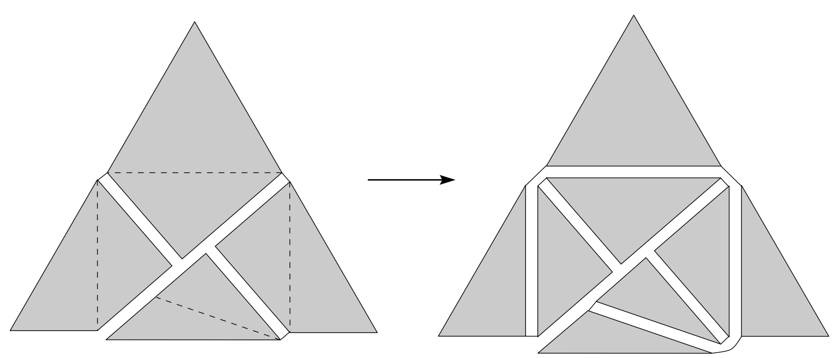

Figure 31: Converting the linearly hinged dissection in Figure 1 into a cyclicly hinged dissection.

If H has n pieces, then when we reduce the hinging structure to a spanning tree it has $n - 1$ hinges. Our construction doubles every original hinge, and adds an additional hinge for every leaf (a piece adjacent to only one original hinge), for a total of at most $3n - 3$. There is one piece per hinge in a cyclicly hinged dissection, so the number of pieces in H' is at most $3n - 3$. □

We are now in the position to show how to hinge-dissect polyforms of different types.

Proof (Theorem 14): Start with the cyclicly hinged dissection from Lemma 15. Now we want to add cuts so that there is a hinge at the midpoint of each edge in P_i for all i. This can be done as follows. For each $i \in \{1, \ldots, k\}$, and for each edge e of P_i that does not already have a hinge at its midpoint, consider the piece Q whose boundary contains e's midpoint when H is rotated into P_i. Refer to Figure 32. Let q_1 and q_2 denote the paths of Q's boundary connecting the two hinges incident to Q, where the paths include their endpoints. Order q_1 and q_2 so that the midpoint of e is on q_1. Pick an arbitrary point r on q_2, add a polygonal cut from r to the midpoint of e, and add a hinge connecting the two pieces at the midpoint of e. Figure 32 shows the special case in which r is chosen to be an endpoint of q_2, i.e., a hinge. In this case, the hinge at r is assigned to the piece of Q that is not incident to the other endpoint of q_2, and the hinged dissection remains cyclic.

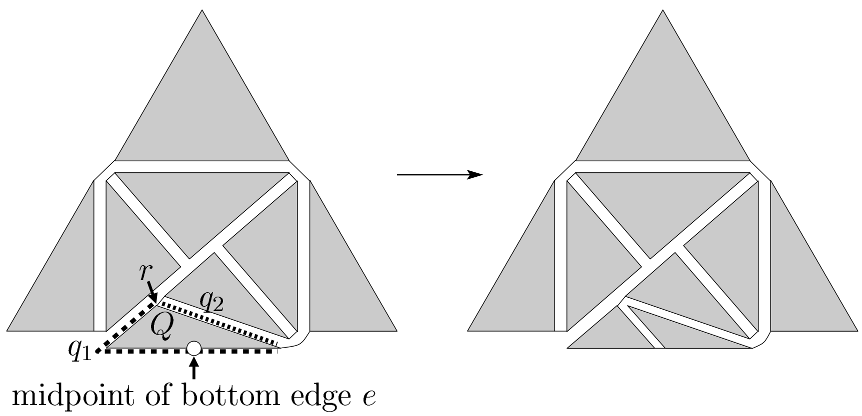

Figure 32: In the cyclicly hinged dissection from Figure 31, cutting piece Q into two pieces so that the midpoint of the triangle's bottom edge e becomes a hinge.

Performing this operation for all choices of i and e, we obtain a cyclicly hinged dissection $\hat{H}$ that can be repeated n times to obtain H'. See Figure 33 for a complete example. Each repetition of $\hat{H}$ can be thought of, in particular, as a subdivision of P_i with hinges at the midpoints. Thus, as we proved in Theorem 13, H' can be rotated into any restricted polyform of type P_i, and this holds for any i.

We started with the $(3n-3)$-piece cyclicly hinged dissection from Lemma 15, and added at most one piece per midpoint of an edge of a polygon P_i. Therefore, we added $\sum_{i=1}^{k} p_i$ pieces, for a total of $3n - 3 + \sum_{i=1}^{k} p_i$. □

7.2 $n \times k$-Regulars to $n \times k'$-Regulars

One particularly interesting application of the previous result is that, provided there is a hinged dissection of a regular k-gon into a regular k'-gon, there is a hinged dissection for each $n \geq 1$ that rotates into all $n \times k$-regulars and all $n \times k'$-regulars. In this section, we explore more efficient hinged dissections for polyiamonds, polyominoes, and polyhexes.

To this end, we will use a more powerful technique for building such a hinged dissection. A linearly hinged dissection H between regular polygons P and Q is called *extendible* if

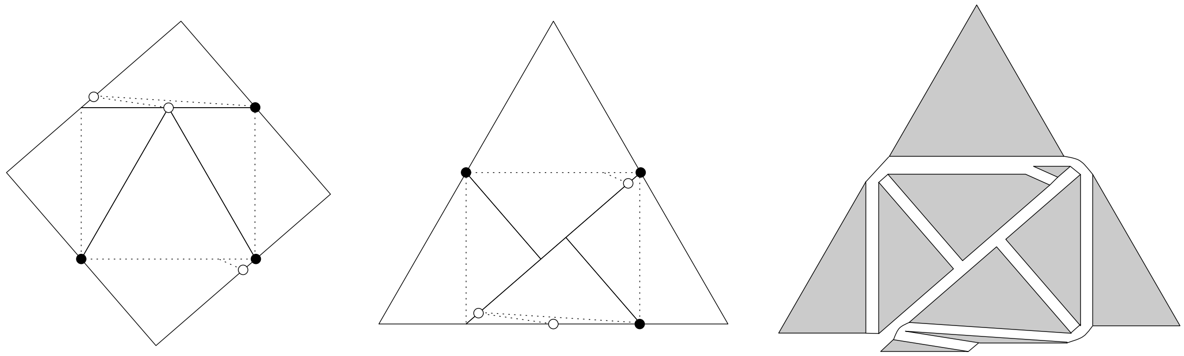

Figure 33: Converting the cyclicly hinged dissection in Figure 31 (roughly) into one with hinges at the midpoints when rotated into either shape. Filled circles are hinges; open circles are midpoints from both shapes.

1. When H is rotated into P, every edge of P has a hinge at an endpoint or at its midpoint.

2. When H is rotated into Q, every edge of Q has a hinge at an endpoint or at its midpoint.

3. There is a vertex in the last piece of the chain and a vertex in the first piece of the chain such that these vertices coincide when H is rotated into P and when H is rotated into Q.

The new flexibility, which will allow us to use fewer pieces in our dissections, is for pieces to be connected by a common vertex instead of just by midpoints. This variation works only because P and Q are regular polygons, and so we can exploit their symmetries. Note also that there is a subtle difference between having a cyclicly hinged dissection and having a chain whose ends coincide: we need the ability to "invert" the chain, which would require twisting a hinge if the two ends were joined together by a hinge.

Theorem 16 *If there is an extendible linearly hinged dissection C between a regular k-gon and a regular k'-gon, then the chain C^n that is formed by concatenating n copies of C rotates into any $n \times k$-regular and into any $n \times k'$-regular, for all $n \geq 1$.*

Proof: Apply Lemma 3 with $c = 1$, for both k and k' symmetrically; we will focus on k. The case $n = 1$ is solved by the given hinged dissection C. Now consider attaching a regular k-gon P to a rotation of C^{n-1} into some $n \times k$-regular R. Let P' be the polygon to which we are attaching P. Let p be a hinge at the midpoint or an endpoint of the edge of P' to which we are attaching P, which is guaranteed to exist because C is extendible. Split C^{n-1} into two chains, $C_p^{n-1,1}$ and $C_p^{n-1,2}$ at p.

Point p is either the midpoint or an endpoint. If it is the midpoint, make cuts in P such that the cuts in $P \cup P'$ exhibit 180°-rotational symmetry about p. Split C at point p into C_p^1 and C_p^2, and splice them together at the original endpoints to give C_p. Then splice C_p into $C_p^{n-1,1}$ and $C_p^{n-1,2}$, and the result is C^n rotated into the desired polyregular $R \cup P$.

If p is an endpoint of the common side of P and P', make cuts in P so that $P \cup P'$ are identical with respect to α-rotational symmetry about p, where α is the interior angle of P. We can then form C^n by cutting and splicing as in the previous case. Again C^n rotates into the desired polyregular $R \cup P$. □

Our goal now is to find efficient extendible linearly hinged dissections between equilateral triangles, squares, and regular hexagons. We start with the first two shapes, by modifying the hinged dissection in Figure 1. We add three more cuts:

1. from the midpoint of the base of the equilateral triangle to the right angle in the small triangular piece,
2. from the midpoint of the base of the equilateral triangle to a point (say, the midpoint) on the left leg of the small triangular piece, and
3. from the right angle of the largest piece to a point near the apex of the equilateral triangle.

These additional cuts produce the extendible linearly hinged dissection in Figure 34.

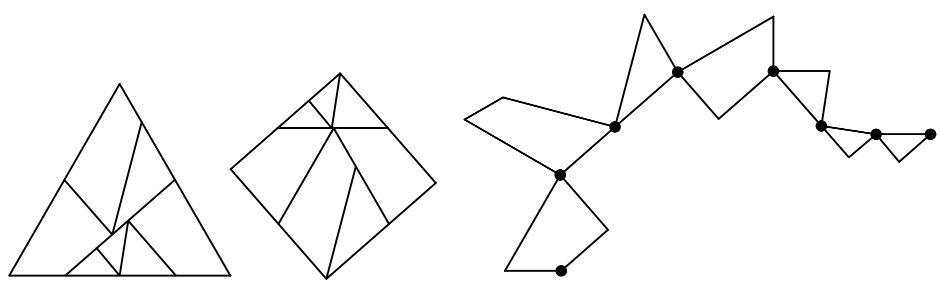

Figure 34: Triangle to a square. (Left) Dissection. (Right) Extendible chain.

Corollary 17 *For any positive n, there is a linearly hinged dissection of $7n$ pieces that rotates into all n-iamonds and all n-ominoes.*

Proof: This follows directly from Theorem 16 and the above construction. □

As an example, we show how to form a tetriamond and a tetromino in Figure 35. There are $7n = 28$ pieces in the chain C^4. We number these pieces in order from 1 to 28. Note that a piece repeats in C^4 after six other pieces, so that, for example, pieces 3, 10, 17, and 24 are identical.

Analogous to the situation in Corollary 6 and Figure 13, we can save on the number of pieces by judicious merging. With respect to the particular tetriamond and tetromino shown, we can merge pieces 22 through 28 together. For any $n > 2$, we can always merge the last seven pieces together. A proof of this, however, requires more care in the ordering of the last several squares (and triangles) chosen in the inductive proof.

Let us next consider n-ominoes and n-hexes. There are several five-piece dissections of a regular hexagon to a square [13, 8], but the best that is known for hinged dissections has

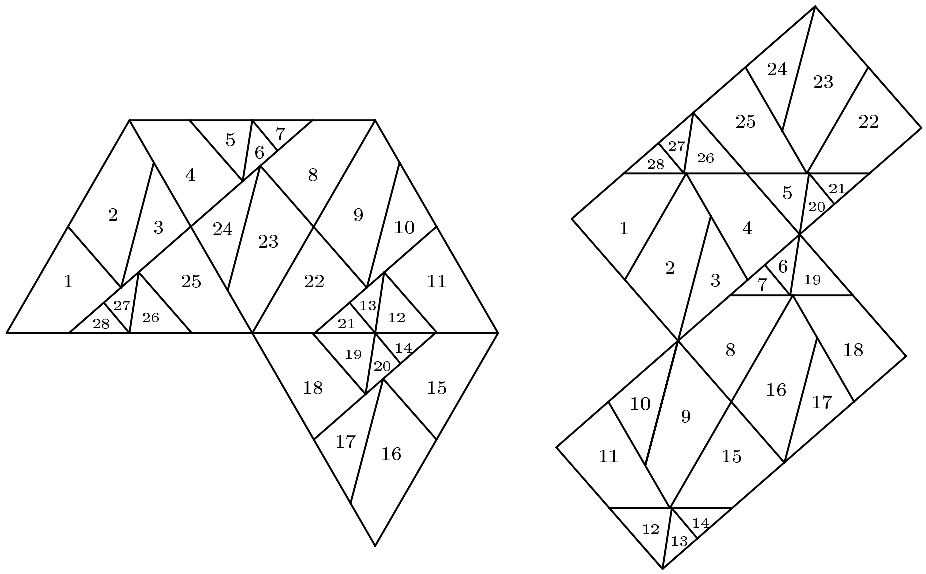

Figure 35: Rotating the chain in Figure 34, repeated four times, into a tetriamond (left) and tetromino (right).

six pieces [9]. One such dissection is linearly hinged, but adapting it to make it extendible in a few number of additional pieces seems difficult.

Thus we start afresh, deriving a TT2-strip dissection by crossposing strips as in Figure 36. (See [8] for a discussion of the T-strip technique.) The hexagon strip consists of halves of hexagons, cut from the midpoint of one side to the midpoint of the opposite side. The square strip consists of rectangles; each is half of a square. The boundaries of the rectangles cross the sides of the hexagons at their midpoints, indicated by the dots.

As discussed in [9], crossing T-strips at midpoints gives rise to hinge points. Other hinge points result from cutting the hexagon and square in half at the midpoints of sides, and placing these halves in the strip so that the resulting vertices touch. This dissection was inspired by an analogous 8-piece hinged dissection of a hexagon to a Greek cross in [9].

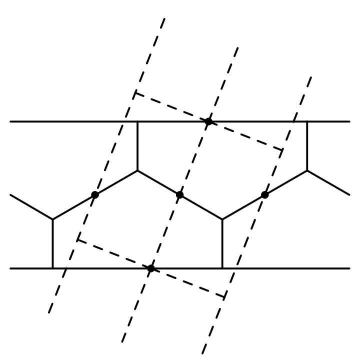

Figure 36: Crossposed strips for a hexagon to a square.

The dissection derived from the crossposition in Figure 36 is cyclicly hingeable but does not have hinges on all six sides of the hexagon. We thus add two additional cuts to produce the dissection on the left of Figure 37. It is cyclicly hinged as shown on the right. Splitting the cycle at any hinge point gives an extendible chain.

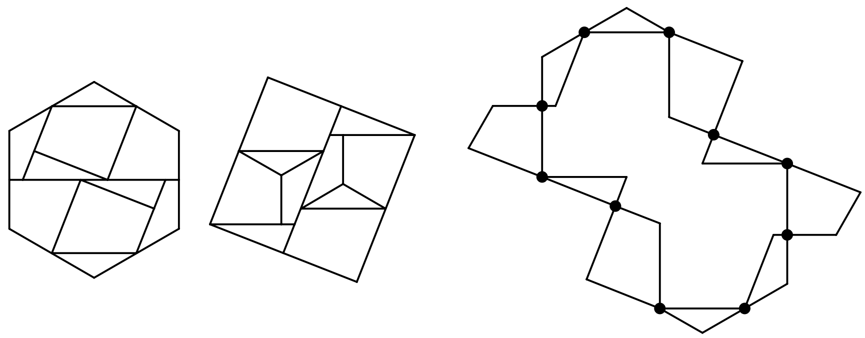

Figure 37: Hexagon to a square. (Left) Dissection. (Right) Extendible cycle.

Corollary 18 *For any positive n, there is a linearly hinged dissection of $10n$ pieces that rotates into all n-ominoes and n-hexes.*

Proof: Again this follows directly from Theorem 16 and the above construction. □

We have not studied how many pieces can be merged together to save a few pieces in the case that $n > 1$.

For handling n-iamonds and n-hexes, we crosspose two strips as in Figure 38. The hexagon strip is created by slicing two isosceles triangles from the hexagon, with each slice going through the midpoints of two of the hexagon's sides. These midpoints are identified with dots. The appropriate angle between the crossposed strips is found by forcing the midpoints of the remaining sides to be positioned on sides of the equilateral triangle. This gives a 6-piece hinged dissection, matching the fewest pieces known for any hinged dissection of an equilateral triangle to a hexagon [9].

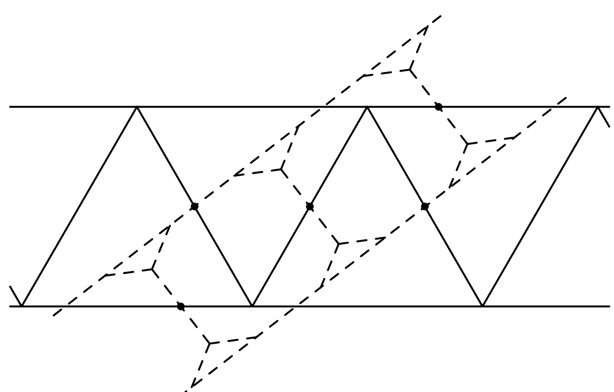

Figure 38: Crossposed strips for an equilateral triangle to a regular hexagon.

To get an extendible linearly hinged dissection, we make three more cuts, giving the dissection in Figure 39. Two of the cuts are through the isosceles triangles, producing a linearly hinged dissection. The third cut is to a vertex of the equilateral triangle, to put a hinge at the base and right side of the equilateral triangle.

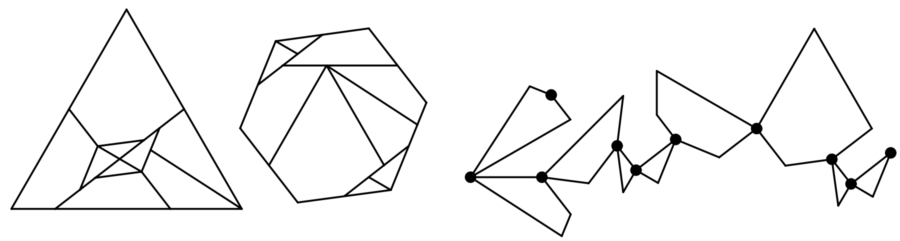

Figure 39: Triangle to hexagon. (Left) Dissection. (Right) Extendible chain.

Corollary 19 *For any positive n, there is a linearly hinged dissection of $9n$ pieces that rotates into all n-iamonds and n-hexes.*

Again, this corollary follows directly from Theorem 16, and we have not studied how many pieces can be merged together to save a few pieces in the case that $n > 1$.

8 Conclusion

Our most general result is that, for any hinged dissection H and $n \geq 1$, there is a hinged dissection H' that rotates into any arrangement of n copies of P joined at corresponding edges, where P is any polygon into which H rotates. In particular, if H is a single-piece "dissection," there is a hinged dissection that rotates into all arrangements of copies of a given polygon P joined at corresponding edges. This statement includes polyregulars as a subclass, for which we showed how to improve the number of pieces. This class contains as subclasses several well studied objects: polyominoes, polyiamonds, and polyhexes. We proved the analogous result for polyabolos (equal-size right isosceles triangles joined edge-to-edge), which do not fall under any of the above classes, but are still considered "polyforms." Using more general (multipiece) dissections for H, we showed how to simultaneously hinge-dissect all polyiamonds and polyominoes; all polyiamonds and polyhexes; and all polyominoes and polyhexes—in general, $n \times k$-regulars and $n \times k'$-regulars when there is a hinged dissection of a regular k-gon into a regular k'-gon.

Following up on our work, Frederickson [9, pp. 234–236] has shown how to obtain similar results for *twist-hinge* dissections, in which hinges cannot be rotated but can be twisted (flipped over 180°). In particular, for $n \geq 5$, he describes a $(4n - 5)$-piece linearly twist-hinged dissection that twists into all n-ominoes. He also describes, for $n \geq 4$, a $6n$-piece linearly twist-hinged dissection that twists into all n-iamonds, and when n is even, into all $(n/2)$-hexes.

Let us conclude with a list of interesting open problems about hinged dissections, focusing on polyforms:

1. Can our results be generalized to arbitrary polyforms, that is, connected edge-to-edge gluings of n nonoverlapping copies of a common polygon?

2. How many pieces are needed for a hinged dissection of all pentominoes? What about general n-ominoes as a function of n? We know of no nontrivial lower bounds.

3. Can any n-omino be hinge-dissected into any m-omino (of an appropriate scale), for all n, m? In Section 5, we proved this is true for $m = 2n$.

4. Can any regular k-gon be hinge-dissected into any regular k'-gon with the same area? Hinged dissections for ten different pairs of regular polygons appear in [9].

5. Is there a single hinged dissection of all $n \times k_1$-regulars, $n \times k_2$-regulars, and $n \times k_3$-regulars? For example, is there a hinged dissection that rotates into all n-iamonds, n-ominoes, and n-hexes? This question is equivalent to asking whether there is a hinged dissection that rotates into an equilateral triangle, square, and regular hexagon.

6. We have shown that there exist rotations of a common hinged dissection into any $n \times k$-regular. Is it possible to continuously rotate the dissection from one configuration to another, while keeping the pieces nonoverlapping? It is known that some hinged dissections cannot be continuously moved in this way [9].

7. It would be interesting to generalize to higher dimensions. For example, *polycubes* are connected face-to-face gluings of nonoverlapping unit (solid) cubes joined face-to-face. Can a collection of solids be hinged together at edges so that the dissection can be rotated into any n-cube (for fixed n)?

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