Packing Unit Squares in Squares: A Survey and New Results

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Abstract

Let s(n) be the side of the smallest square into which we can pack n unit squares. We present a history of this problem, and give the best known upper and lower bounds for s(n) for n≤100, including the best known packings. We also give relatively simple proofs for the values of s(n) when n = 2, 3, 5, 8, 15, 24, and 35, and more complicated proofs for n=7 and 14. We also prove many other lower bounds for various s(n).

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1 Introduction

The problem of packing equal circles in a square has been around for some 40 years and has seen much recent progress [2]. The problem of packing equal squares in a square is only recently becoming well known. Results were less plentiful, as the computer-aided methods available for circles did not generalize for squares, until recently when an effective algorithm was found [20]. We give a few packings which improve upon those in the literature, illustrate a technique for obtaining lower bounds, and exhibit the best known packings for less than one hundred squares.

Let $s(n)$ be the side of the smallest square into which we can pack n unit squares. It is clear that $\sqrt{n} \le s(n) \le \lceil \sqrt{n} \rceil$, the first inequality coming from area considerations, and the second coming from the facts that $s(n)$ is non-decreasing and $s(n^2)=n$. It is not hard to show that $s(2)=s(3)=2$. It is a little harder to show that $s(5)=2+1/\sqrt{2}$ [7].

The number of claims far outweighs the number of published results in this area. Göbel says that Schrijver claims that Bajmóoczy proved $s(7)=s(8)=3$ [7]. Walter Stromquist claimed to have proved $s(6)=3$ and $s(10)=3+1/\sqrt{2}$, and claimed to know how to prove $s(14)=s(15)=4$ and $s(24)=5$ [13]. Trevor Green sent me a proof for $s(6)=3$. None of these proofs were published. Said El Moumni evidently proved $s(7)=s(8)=3$ and $s(15)=4$ [12] but no one was aware of this until recently. Finally, in 2002, Kearney and Shiu published a proof of $s(6)=3$ [9].

In 2003, Stromquist proved $s(10)=3+1/\sqrt{2}$ [18]. In 2005, Nagamochi proved that $s(n^2-2)=s(n^2-1)=n$ [19]. Also in 2005, Thierry Gensane and Philippe Ryckelynck published an inflation algorithm for finding good packings and found the first computer packing that might be optimal. [20]. There are many other good packings thought to be optimal, but as of yet no proofs. Here we prove the values of $s(n)$ for square n and n=2, 3, 5, 7, 8, 14, 15, 24, and 35.

Previous results can be found in Section 2. Reccent packings appear in Section 3. In Section 4 we prove some technical lemmas that we use in Section 5 to prove the values of $s(n)$ mentioned above. Lists of the best known upper and lower bounds for $s(n)$ are given in the Appendix. Many of the results given are taken from unpublished letters and manuscripts, and private communications.

2 Previous Results

Göbel was the first to publish on the subject [7]. He found that $a^2+a+3+\lfloor (a-1)\sqrt{2} \rfloor$ squares can be packed in a square of side $a+1+1/\sqrt{2}$ by placing a diagonal strip of squares at a 45° angle. This gives the best known packings for all values of a except for a=3 and a=6 (see Figure 1).
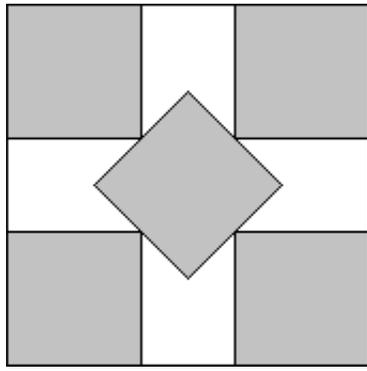
$s(5)=2+1/\sqrt{2}$
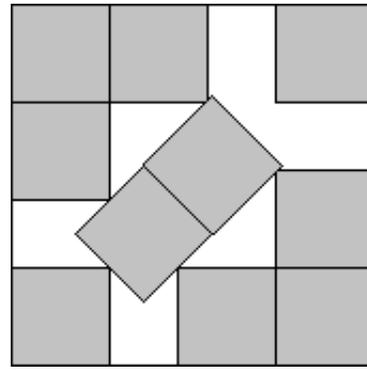
$s(10)=3+1/\sqrt{2}$
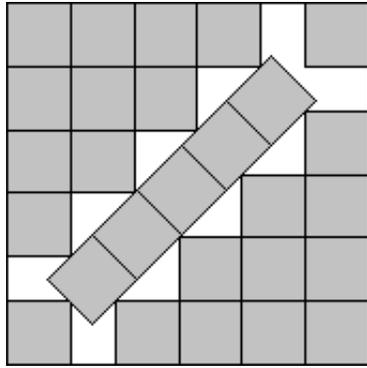
$s(27)\leq5+1/\sqrt{2}$
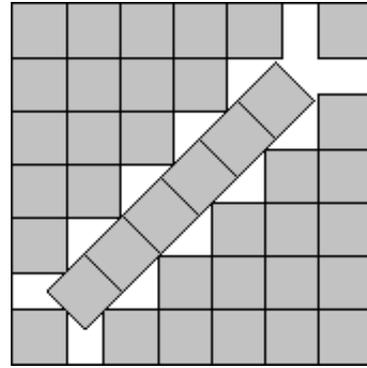
$s(38)\leq6+1/\sqrt{2}$
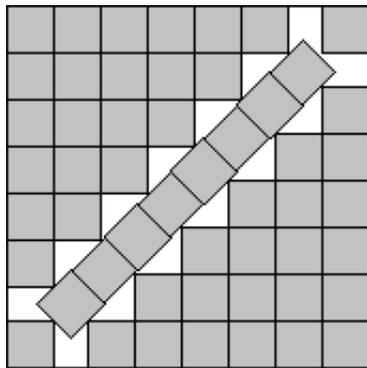
$s(52)\leq7+1/\sqrt{2}$
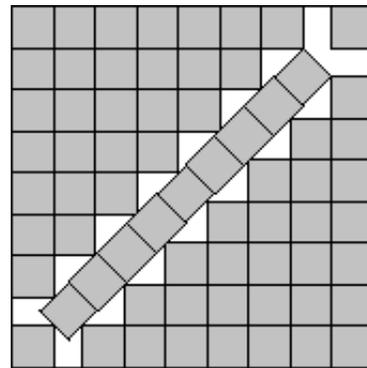
$s(67)\leq8+1/\sqrt{2}$
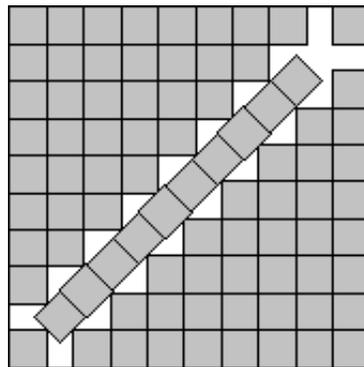
$s(84)\leq9+1/\sqrt{2}$

Figure 1.

By unrotating some rotated squares in the corner, we get some alternate packings for n=10, 67, and 84. (see Figure 2). David Cantell noticed in 2005 that alternative packings exist for n=27, 38, 52, and 84 using a minimum number of rotated squares (see Figure 2) [17].

$s(10)=3+1/\sqrt{2}$ $s(67)\leq8+1/\sqrt{2}$

$s(84)\leq9+1/\sqrt{2}$ $s(84)\leq9+1/\sqrt{2}$

$s(27)\leq5+1/\sqrt{2}$ $s(38)\leq6+1/\sqrt{2}$

$$s(52)\leq 7+1/\sqrt{2} \qquad\qquad s(84)\leq 9+1/\sqrt{2}$$
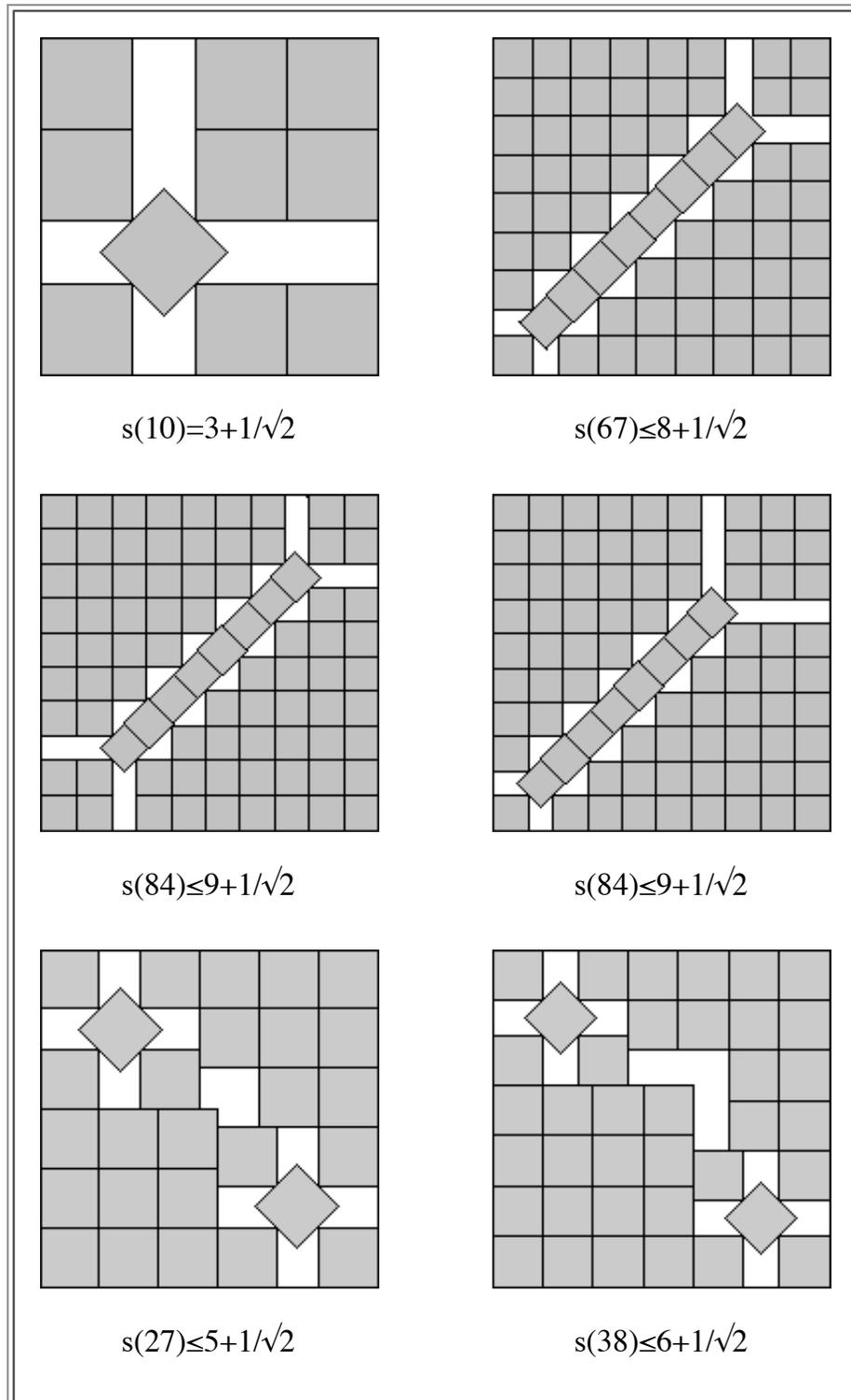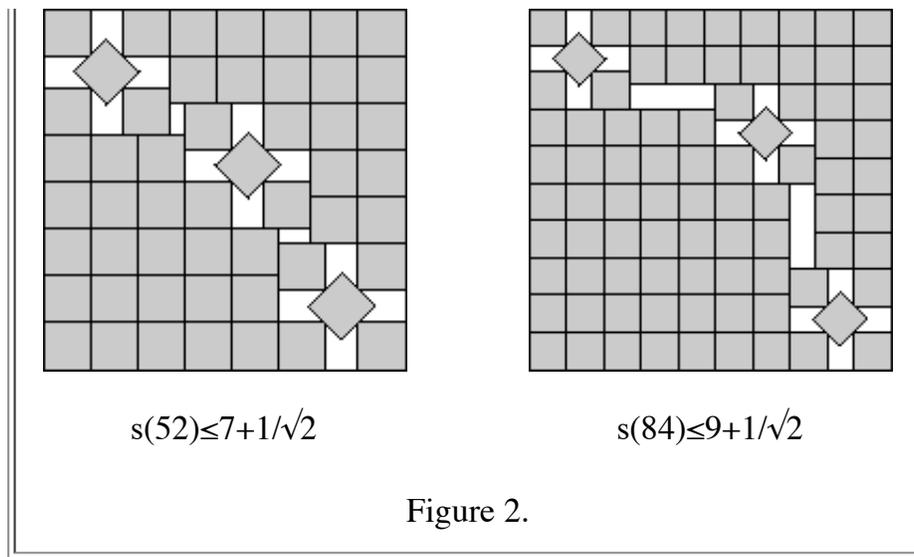
Figure 2.

It is clear that $n+2\lfloor s(n)\rfloor+1$ squares can be packed in a square of side $s(n)+1$ by packing n squares inside a square of side $s(n)$ and putting the other squares in an "L" around it. The first four packings in Figure 2 are of this form. Packings not containing an "L" of squares we will call *primitive* packings. From now on, we will only illustrate primitive packings.

Göbel also found that if integers a and b satisfied $a-1<b/\sqrt{2}<a+1$, then $2a^2+2a+b^2$ squares can be packed inside a square of side $a+1+b/\sqrt{2}$. This is accomplished by placing a $b\times b$ square of squares at a 45° angle in the center. This gives the best known packings for 28, 40, 65, and 89 squares (see Figure 3). Adding an "L" around the packing of 65 squares gives the best known packing of 82 squares. The packing for n=40 is rigid.

$$s(28)\leq 3+2\sqrt{2} \qquad\qquad s(40)\leq 4+2\sqrt{2}$$
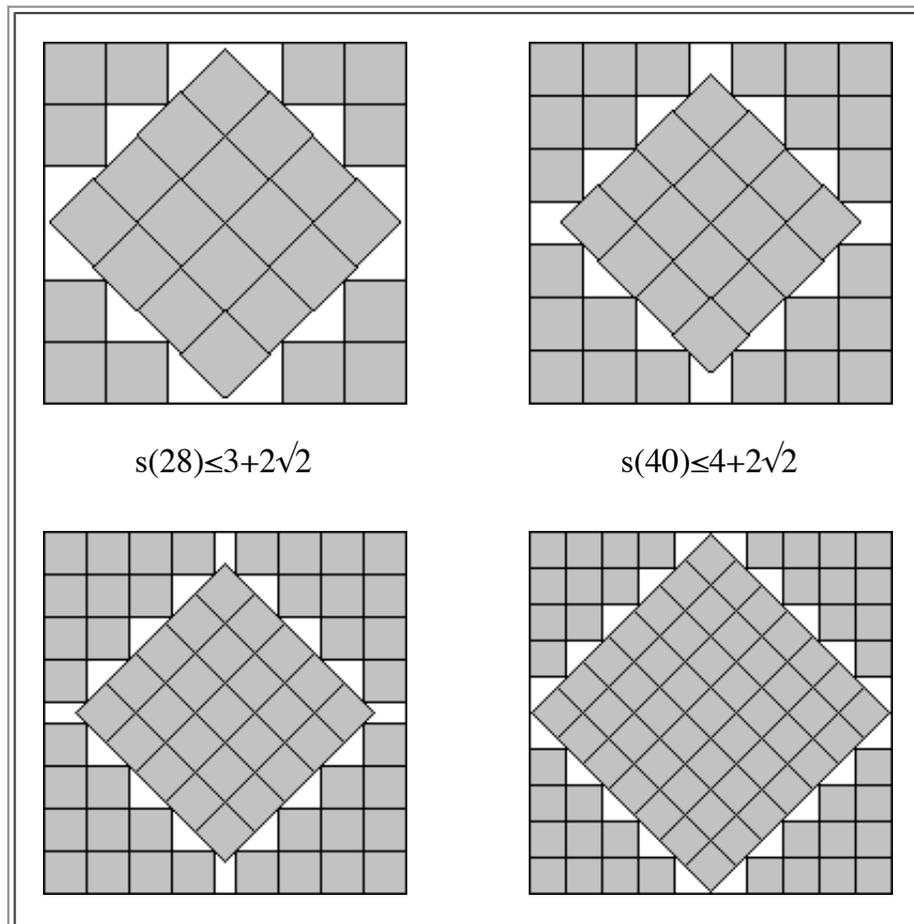
Charles Cottingham, who improved some of Göbel's packings for n≤49, was the first to use diagonal strips of width 2 [6]. Soon after he produced a packing of 19 squares with a diagonal strip of width 2, Robert Wainwright improved Cottingham's packing slightly (see Figure 4) [4]. In 2002, David Cantrell found some alternative packings for 19 squares [14].

$s(19) \leq 3 + 4\sqrt{2}/3$
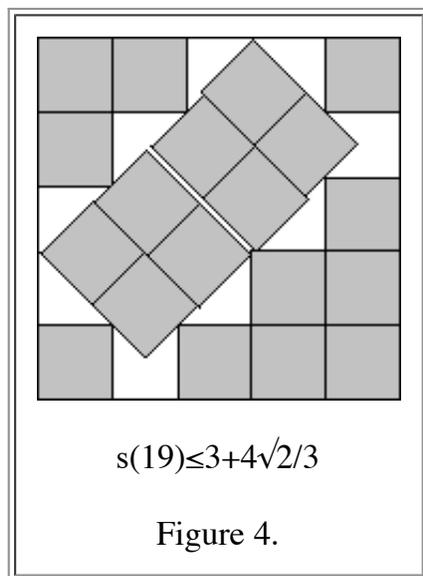
Figure 4.

In 1980, Evert Stenlund improved many of Cottingham's packings, and provided packings for n≤100 [6]. His packing of 66 squares uses a diagonal strip of width 3 (see Figure 5). In this packing, the diagonal squares touch only the squares in the upper right and lower left corners. Adding an "L" to this packing gives the best known packing of 83 squares.

$s(66) \leq 3 + 4\sqrt{2}$
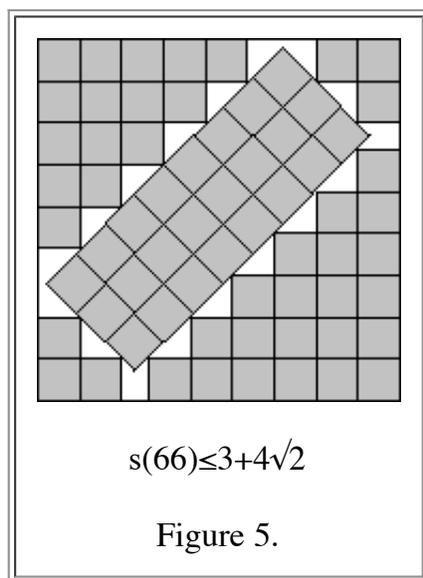
Figure 5.

The best known packings for many values of n are more complicated. Many seem to require packing with squares at angles other than 0° and 45°. In 1979, Walter Trump improved Göbel's packing of 11 squares (see Figure 6). Many people have independently discovered this packing. The original discovery has been incorrectly attributed to Gustafson and Thule [11]. The middle squares are tilted about 40.182^o, and there is a small gap between these squares. This packing is also rigid.

s(11)<3.8771
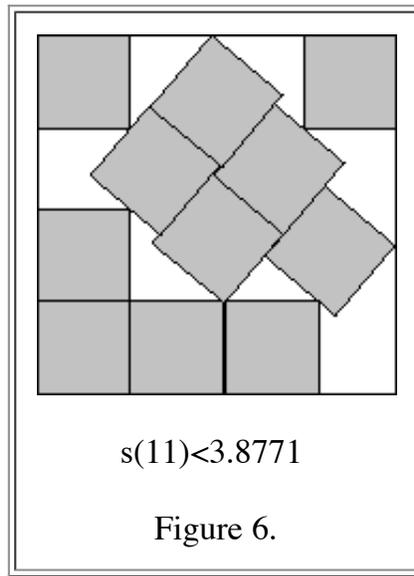
Figure 6.

In 1980, Hämäläinen improved on Göbel's packing of 18 squares (see Figure 7) [6]. In 1981, Mats Gustafson found an alternative optimal packing of 18 squares (see Figure 7). The middle squares in these packings are tilted by an angle of arcsin(($\sqrt{7}$-1)/4)≅24.295º. In 2002, David Cantrell found another alternative packing (see Figure 7) [14] that is useful in building the best known packing for n=68 (see Figure 11). In 2004, the computer program of Gensane and Ryckelynck found yet another alternative packing (see Figure 7) [20].

s(18)≤(7+$\sqrt{7}$)/2

s(18)≤(7+$\sqrt{7}$)/2

s(18)≤(7+$\sqrt{7}$)/2

s(18)≤(7+$\sqrt{7}$)/2

Figure 7.
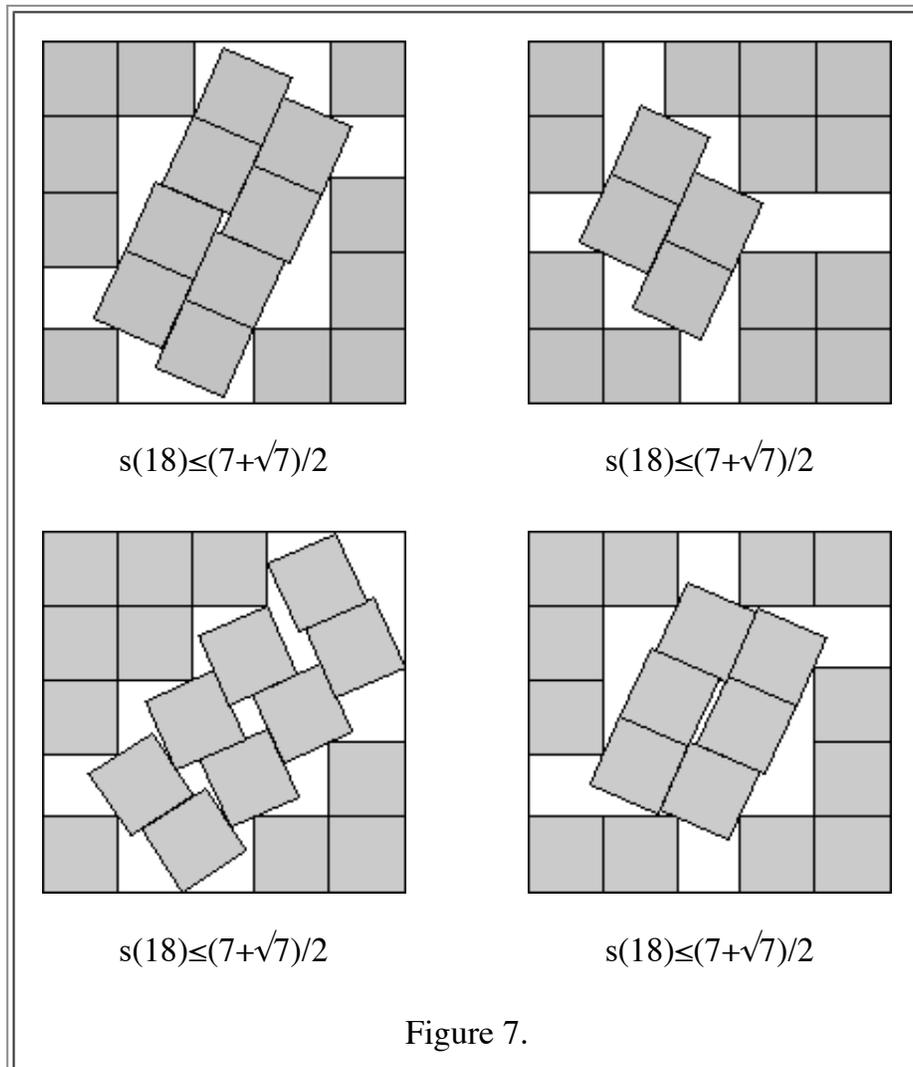
In [3], Erdös and Graham define $W(s)=s^2-\max\{n:s(n)\leq s\}$. Thus $W(s)$ is the wasted area in the optimal packing of unit squares into an $s \times s$ square. They show (by constructing explicit packings) that $W(s)=O(s^{7/11})$. In [10], it is mentioned that Montgomery has improved this result to $W(s)=O(s^{3-\sqrt{3}/2+\varepsilon})$ for every $\varepsilon>0$.

In [10], Roth and Vaughan establish a non-trivial lower bound for $W(s)$. They show that if $s(s-\lfloor s\rfloor)>1/6$, then $W(s)\geq 10^{-100}\sqrt{(s\mid s-\lfloor s+1/2\rfloor\mid)}$. This implies that $W(s)$ is not $O(s^{\alpha})$ when $\alpha<1/2$.

It was conjectured that $s(n^2-n)=n$ whenever n is small. The smallest known counterexample of this conjecture, due to Lars Cleemann, is $s(17^2-17)<17$. That is, 272 squares can be packed into a square of side 17 in such a way that the the square can be squeezed together slightly (see Figure 8). Three squares are tilted by an angle of 45°, and the other tilted squares are tilted by an angle of arctan(8/15).

s(272)<17
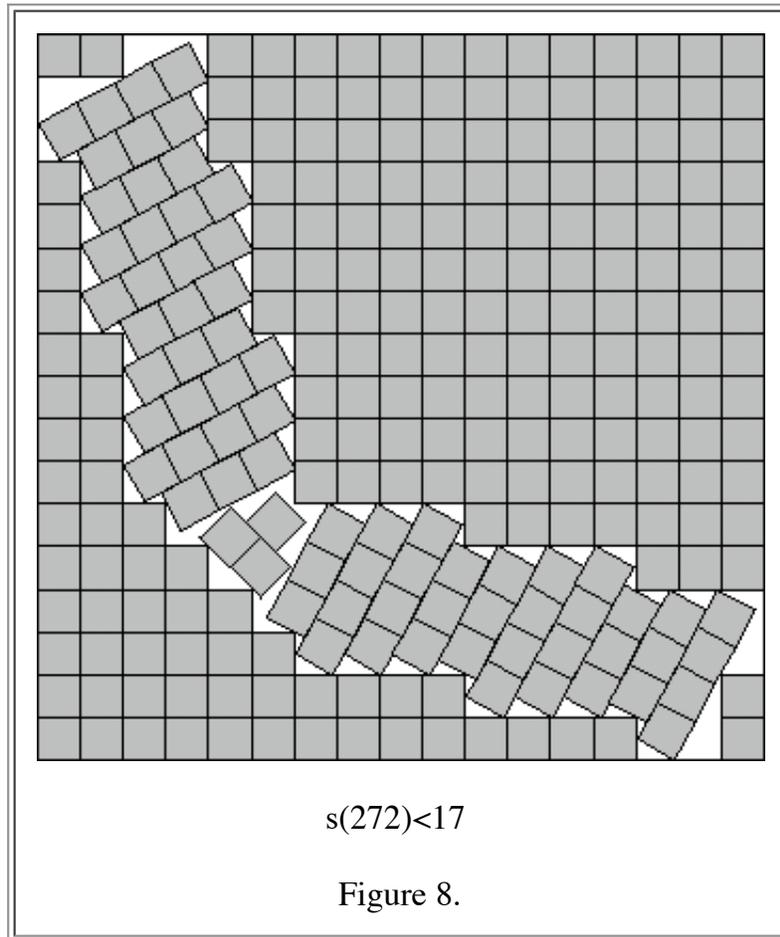
Figure 8.

3 New Packings

We can generalize the packings in Figure 3 by placing the central square a little off center. We can pack $2a^2+2a+b^2$ squares in a rectangle with sides $a+1/2+b/\sqrt{2}$ and $a+3/2+b/\sqrt{2}$. Adding a column of squares to the side of this, we get a packing of $2a^2+4a+b^2+1$ squares in a square of side $a+3/2+b/\sqrt{2}$. This gives the best known packings for 26 and 85 squares (see Figure 9).

$$s(26)\leq(7+3\sqrt{2})/2 \qquad s(85)\leq11/2+3\sqrt{2}$$

Figure 9.
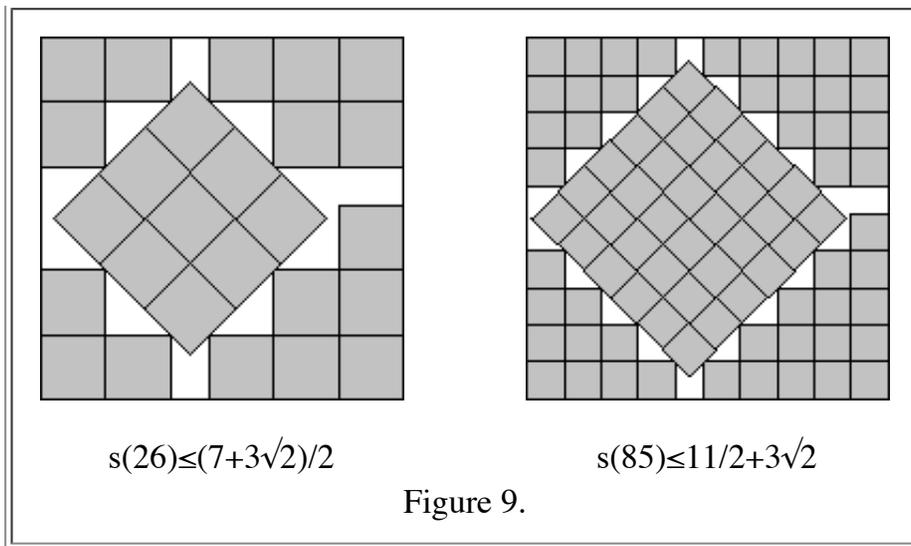
Note that a diagonal strip of width 2 or 3 must be off center in order to be optimal. Otherwise one could place at least as many squares by not rotating them.

Stenlund also modified a diagonal strip of width 4 to pack 87 squares. In 2002, David Cantrell changed the angles slightly to give a minutely better packing (see Figure 10) [14]. There is a thin space between two of the diagonal strips. Compare this with the packing of 19 squares in Figure 4.

$$s(87)<9.8520$$
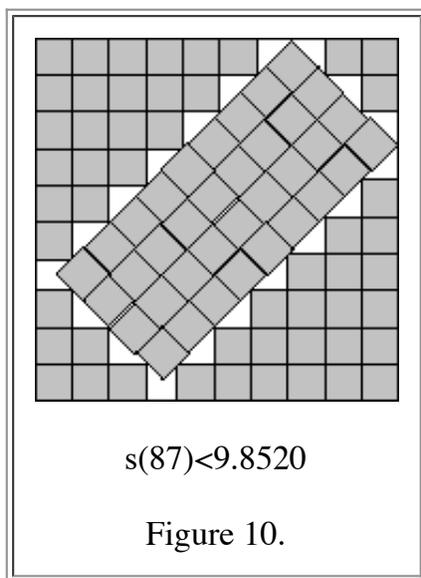
Figure 10.

In 2002, David Cantrell found packings of 53 and 68 squares that are slightly better than Göbel's (see Figure 11) [14].

s(53)<7.8231

s(68)≤15/2+√7/2
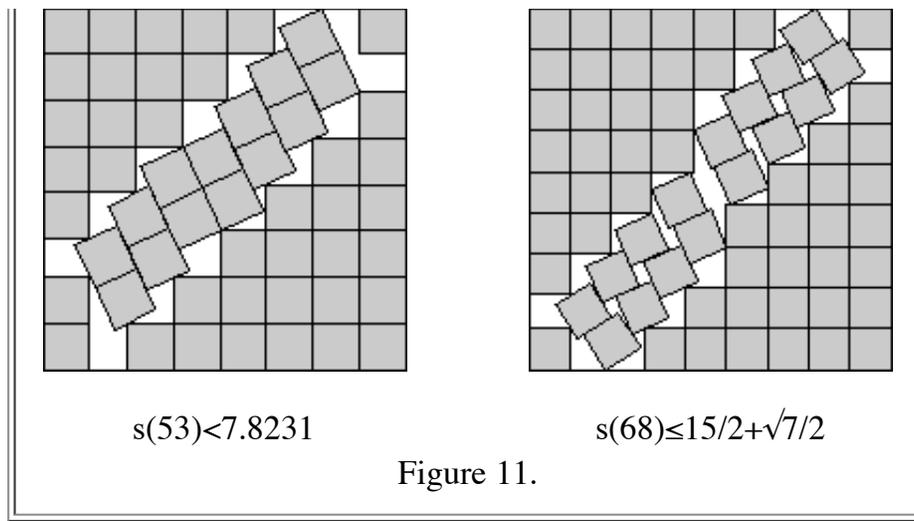
Figure 11.

In 1980, Pertti Hämäläinen improved Göbel's packing of 17 squares using a different arrangement of squares at a 45° angle. But in 1998, John Bidwell, an undergraduate student at the University of Hawaii, improved this packing (see Figure 12) [1]. It is the smallest example where the best known packing contains squares at three different angles.

Also in 1998, I improved the best known packing of 29 squares using a modified diagonal strip of width 2. A few months later, Bidwell improved my packing slightly [1]. In 2005, David Cantrell used a completely different idea to find a better packing [17]. But in 2004, the computer program of Thierry Gensane and Philippe Ryckelynck found what now stands as the best known packing for 29 squares (see Figure 12) [20]. This packing contains squares at no less than 6 different angles.

s(17)<4.6756

s(29)<5.9344
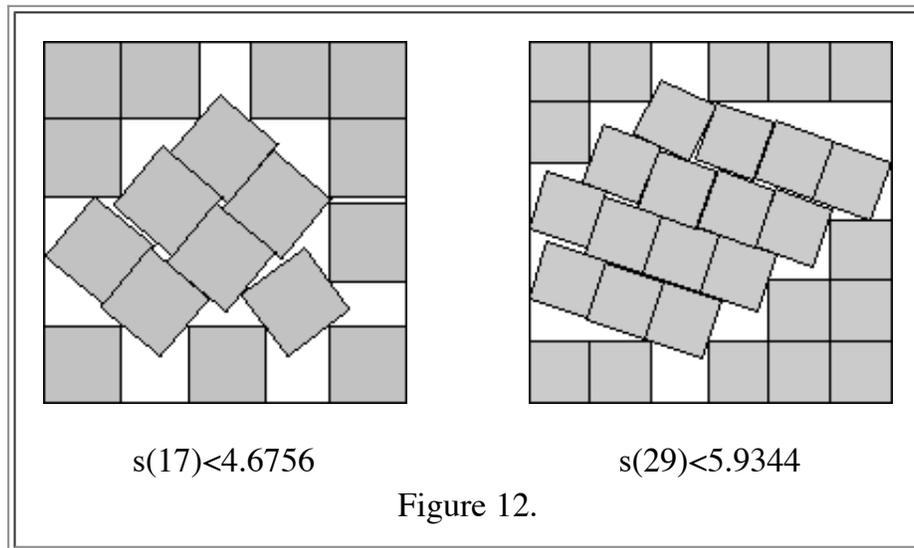
Figure 12.

In 1997, I generalized Stenlund's packing of 41 squares in Figure 4 to give packings of 70 and 88 squares using strips of width 2. In 2002, David Cantrell modified these packings to give the best known packings (see Figure 13) [14].

s(70)<8.9121   s(88)<9.9018
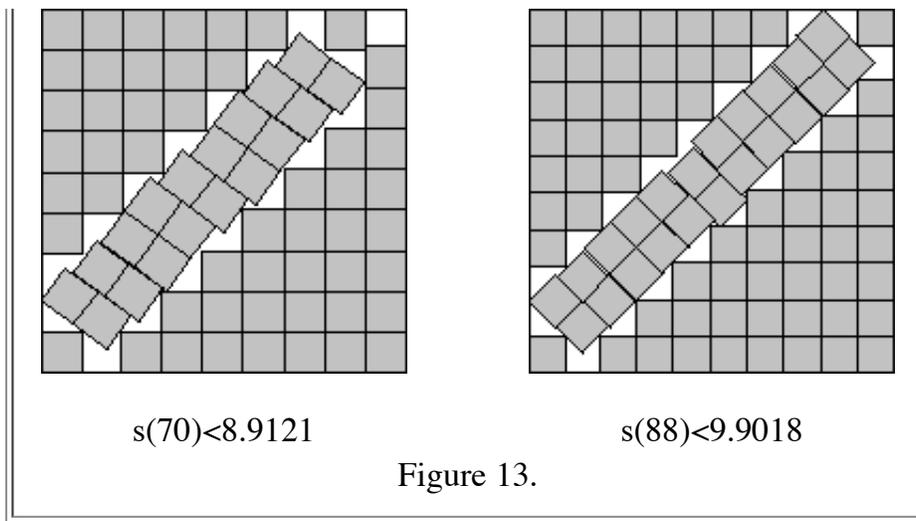
Figure 13.

In 1997, I modified a diagonal strip of 2 squares to get a packing of 54 squares based on Wainwright's packing of 19 squares in Figure 4. In 2002, David Cantrell used his alternative packings of 19 squares to improve the packing for 54 squares [14]. He further improved this packing in 2005 (see Figure 14) [17].

In 1997, I modified a diagonal strip of 4 squares to get a packing of 69 squares. In 2002, David Cantrell improved this to produce the best known packing of 69 squares (see Figure 14) [14].

s(54)<7.8488   s(69)<8.8562
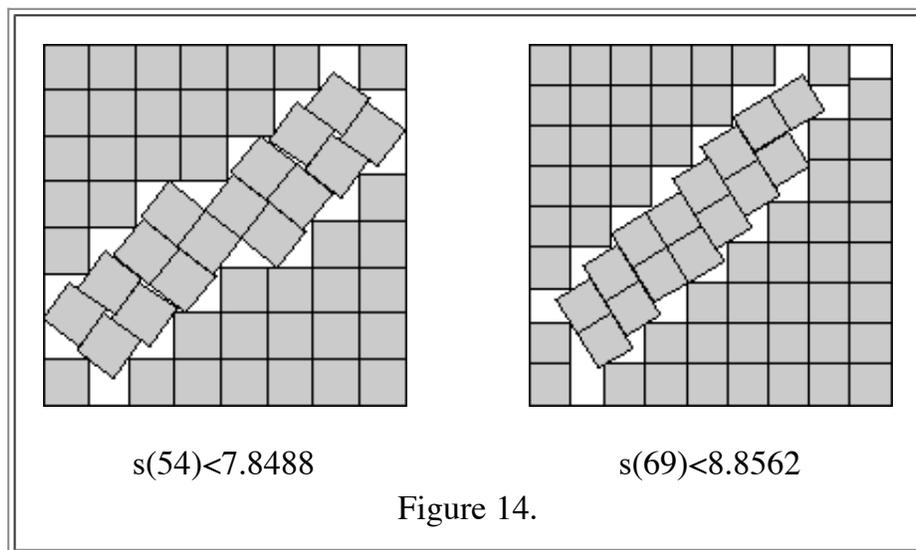
Figure 14.

In 2002, David Cantrell used a completely new idea to find the best known packing of 39 squares (see Figure 15) [14].

We can generalize the packings in Figure 7 to provide the best known packings of 86 squares (see Figure 15). The angle of the tilted squares is the same as in that Figure.

s(39)<6.8189   s(86)≤(17+√7)/2
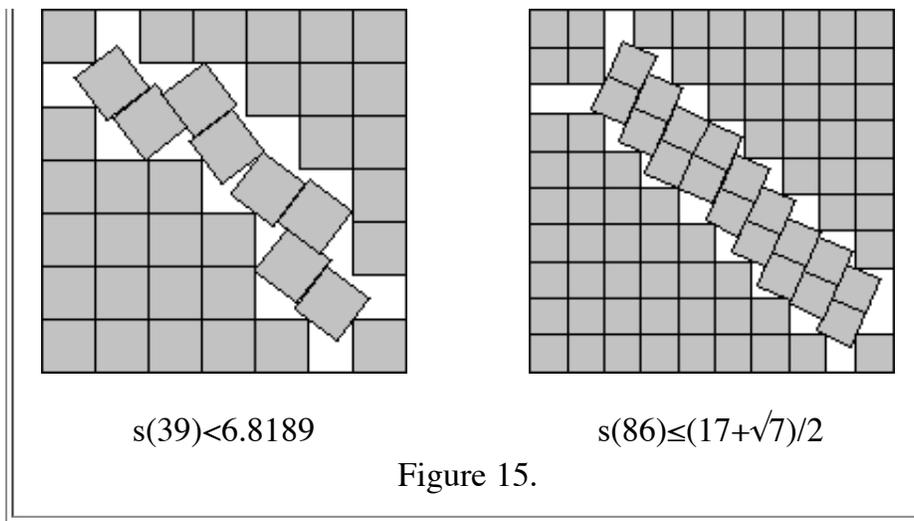
Figure 15.

In 1997, I improved the packing for 37 squares using a modified diagonal strip of width 3. In 2002, David Cantrell improved this slightly (see Figure 16) [14]. The slanted squares are tilted at angles of approximately 42.086º and 45º. Adding an ``L" to this packing of 37 squares gives the best known packing of 50 squares.

In 1979, Charles Cottingham found a packing of 41 squares that was only improved in 2005 by David Cantrell (see Figure 16). [17]

In 2002, David Cantrell found the first packing of 55 squares in a square of side less than 8 [14]. He then improved this packing in 2005. (see Figure 16) [17].

Also in 2005, David Cantrell found the first packing of 71 squares in a square of side less than 9. (see Figure 16) [17].

In 2009, Károly Hajba found the first known packing of 51 squares in a square of side less than $7+1/\sqrt{2}$. (see Figure 16) [16].

s(37)<6.5987   s(41)<6.9473

s(55)<7.9871

s(71)<8.9633

s(51)<7.7044

Figure 16.
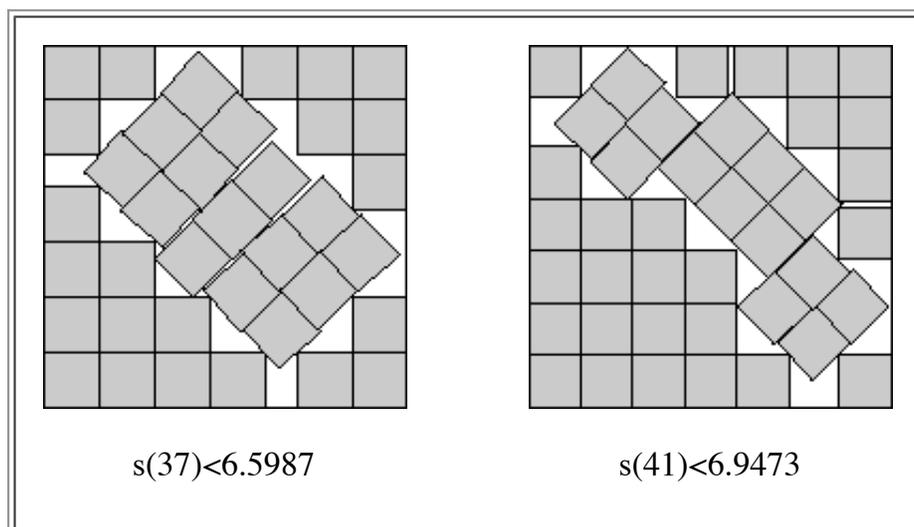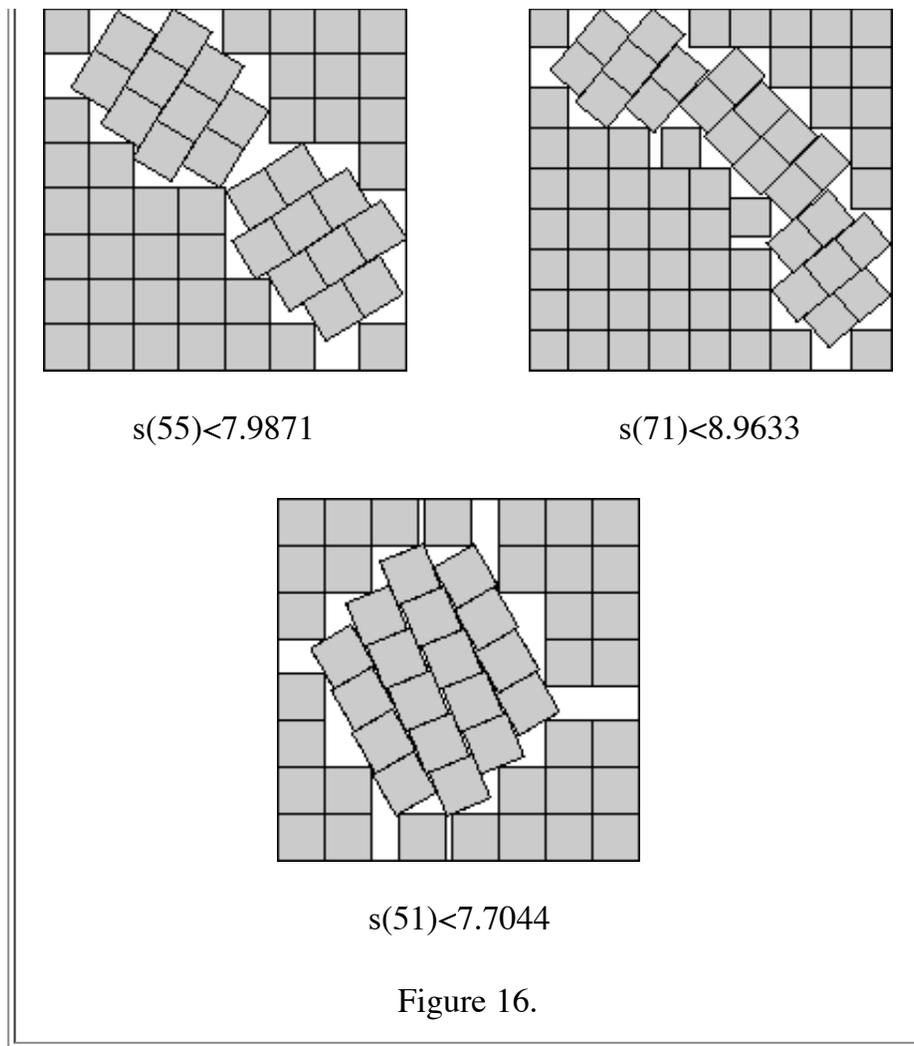
Finally, we make the following conjecture:

Conjecture 1. If $s(n^2-k)=n$, then $s((n+1)^2-k)=n+1$.

That is, if omitting k squares from an n × n square does not admit a smaller packing, then the same will be true for omitting k squares from any larger perfect square packing. This is true of all the best known packings.

4 Technical Lemmas

Lemma 1. Any unit square inside the first quadrant whose center is in $[0,1]^2$ contains the point (1,1).

Proof: It suffices to show that a unit square in the first quadrant that touches the x-axis and y-axis contains the point (1,1). If the square is at an angle θ, it contains the points $(\sin\theta, 0)$ and $(0, \cos\theta)$ (see Figure 17). The two other corners of the square, $(\cos\theta, \cos\theta+\sin\theta)$ and $(\cos\theta+\sin\theta, \sin\theta)$, lie on the line $y - \sin\theta = -\cot\theta (x - \sin\theta - \cos\theta)$. In particular, when x=1,

$$y = [\sin^2\theta - \cos\theta (1 - \sin\theta - \cos\theta)] / \sin\theta = [(1 - \sin\theta) (1 - \cos\theta) + \sin\theta] / \sin\theta \geq 1. \blacksquare$$
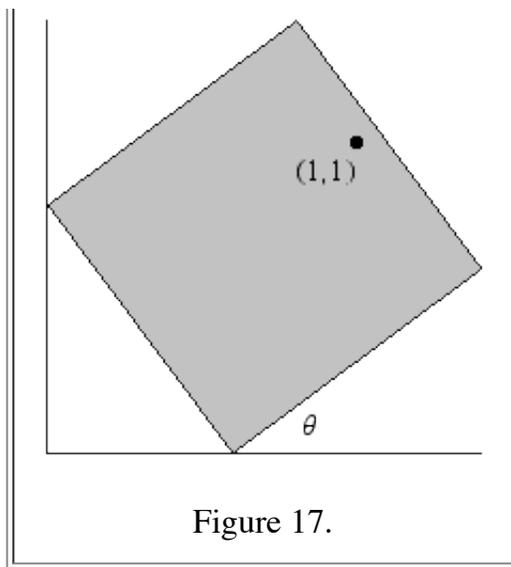
Figure 17.

Lemma 2. Let $0<x\le1$, $0<y\le1$, and $x+2y<2\sqrt{2}$. Then any unit square inside the first quadrant whose center is contained in $[1,1+x]\times[0,y]$ contains either the point $(1,y)$ or the point $(1+x,y)$.

Proof: It suffices to show that a unit square u whose center is contained in $[1,1+x]\times[0,y]$ that contains the points $(1,y)$ and $(1+x,y)$ on its boundary contains a point on the x-axis. This is true if $(1,y)$ and $(1+x,y)$ lie on the same side of u. If u is at an angle θ, then the lowest corner of the square is

$$(1 + x + (1 - x\sin\theta)\sin\theta - \cos\theta , y - (1 - x\sin\theta)\cos\theta - \sin\theta)$$

(see Figure 18). This point lies outside the first quadrant when $f(\theta) = \cos\theta + \sin\theta - x\sin\theta\cos\theta > y$. Since

$$f'(\theta) = (\cos\theta - \sin\theta)[1 - x(\cos\theta + \sin\theta)],$$

the critical points of $f(\theta)$ are

$$(\cos\theta,\sin\theta)=(1/\sqrt{2},1/\sqrt{2}) \text{ and } (\cos\theta,\sin\theta) = (1 \pm \sqrt{(2x^2-1)} / 2x , 1 \mp \sqrt{(2x^2-1)} / 2x).$$

Checking these 3 values and the endpoints, the global minimum of $f(\theta)$ occurs at $\theta=45^\circ$. Therefore, when $y<\sqrt{2}-x/2$, u contains some point of the x-axis. ∎
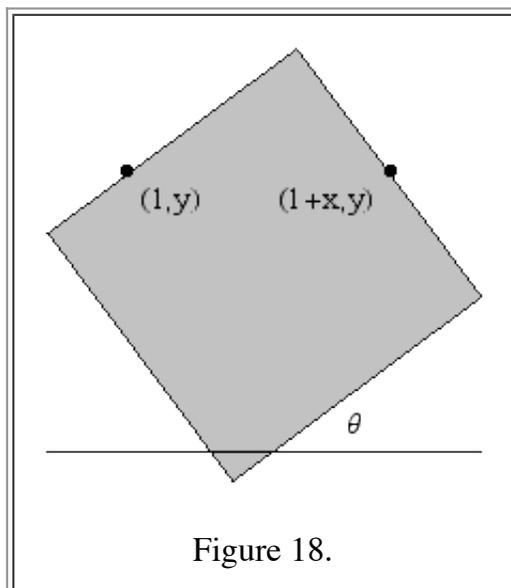
Figure 18.

Lemma 3. If the center of a unit square u is contained in $\triangle ABC$, and each side of the triangle has length no more than 1, then u contains A, B, or C.

Proof: The diagonals of u divide the plane into 4 regions, labeled clockwise as R_1, R_2, R_3, and R_4 (see Figure 19). These regions are closed, and intersect only on the diagonals. The points A, B, and C cannot all be on one side of either one of these diagonals, for then $\triangle ABC$ would not contain the center of u. Thus either both R_1 and R_3 contain vertices of the triangle, or both R_2 and R_4 do. In either case, two vertices of $\triangle$ ABC are closest to two opposite sides of u. Since the distance between these vertices is no more than 1, u must contain at least one of these points. ■

Figure 19.
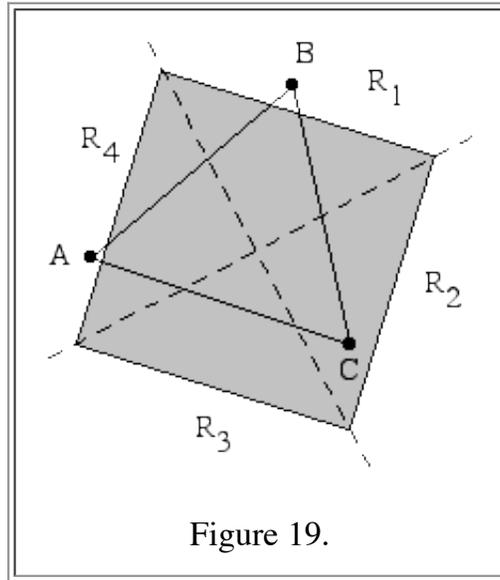
Lemma 4. If the center of a unit square u is contained in the rectangle R=[0,1]x[0,.4], then u contains a vertex of R.

Proof: Let A=(0,0), B=(0,.4), C=(1,0), and D=(1,.4). It suffices to show that any u that contains A and B on its boundary and whose center is in R contains either C or D (see Figure 20). This is clearly the case if A and B lie on the same side of u. When $\theta=45°$, u contains both C and D. It is easy to see that when $\theta<45°$, u contains D, and when $\theta>45°$, u contains C. ■

Figure 20.
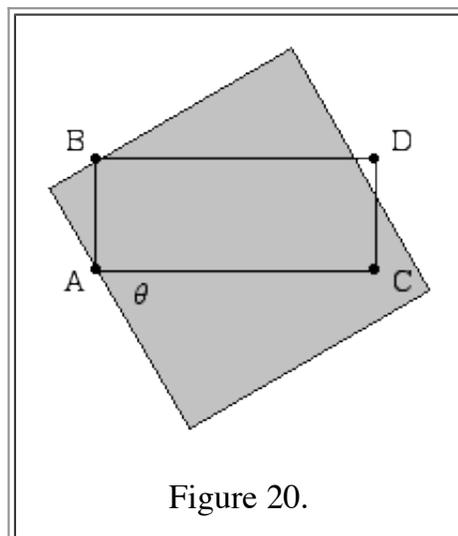
Lemma 5. If a unit square has its center below the line y=1, and is entirely above the x-axis, then the length of the intersection of the line y=1 with the square is at least $2\sqrt{2}$-2.

Proof: Let L be the line y=1. Since the center of the square is below L, 2 or fewer corners of the square are above L. If 2 corners are above L, then L intersects 2 opposite sides of the square, and therefore the intersection has length at least 1. If none of the corners are above L, 2 corners sit on the x-axis, and the length of the intersection is exactly 1. We therefore assume 1 corner is above L. In this case, the intersection is made smaller by moving the square downwards until one of the corners is touching the x-axis.

If the square makes an angle θ with the x-axis, the vertical line segment in Figure 21 has length $(\sin\theta + \cos\theta - 1)$, so the length of the intersection with L is

$$D = (\sin\theta + \cos\theta - 1)(\tan\theta + \cot\theta) = (\sin\theta + \cos\theta - 1) / (\sin\theta \cos\theta).$$

This is minimized when $dD/d\theta = (\sin\theta - \cos\theta)(1 - \cos\theta)(1 - \sin\theta) / (\cos^2\theta \sin^2\theta) = 0$, which occurs at θ=45°. Thus $D \geq 2\sqrt{2}$-2. ∎

Figure 21.
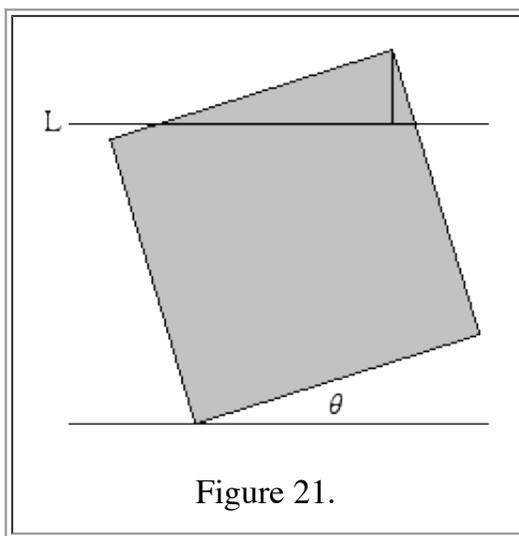
Lemma 6. If a unit square has its center in the region $[0,1]^2$, does not contain either of the points (0,1) and (1,1), and is entirely above the x-axis, then the square covers some point (0,y) for $1/2 \leq y \leq 1$ and some point (1,y) for $1/2 \leq y \leq 1$.

Proof: Lowering the square until it touches the x-axis lowers any points of intersection with the y-axis. Moving the square right until it touches the point (1,1) makes any intersection with the y-axis smaller. We will show that any such square touching the x-axis and the point (1,1) covers some point (0,y) for $1/2 \leq y \leq 1$. The rest of the lemma follows from symmetry.

If x is the distance in Figure 22, then $\sin\theta + x\cos\theta = 1$, or $x = (1 - \sin\theta) / \cos\theta$. This means the x-intercept of the square is

$$1 + x\sin\theta - \cos\theta = 1 + \tan\theta (1 - \sin\theta) - \cos\theta.$$

Therefore the left corner of the square is $(1 + \tan\theta (1 - \sin\theta) - \cos\theta - \sin\theta , \cos\theta)$. This means the largest y-intercept of the square is

$$D = \cos\theta - \tan\theta (1 + \tan\theta (1 - \sin\theta) - \cos\theta - \sin\theta) = (\cos^3\theta + \sin\theta (1 - \cos\theta)(1 - \sin\theta)) / \cos^2\theta.$$

Since $dD/d\theta = (1 - \cos\theta - \sin\theta)(1 - \sin\theta)/\cos^3\theta < 0$, the minimum value of D is the limit of D as θ approaches 90°, which is 1/2 by L' Hôpital's Rule. ∎
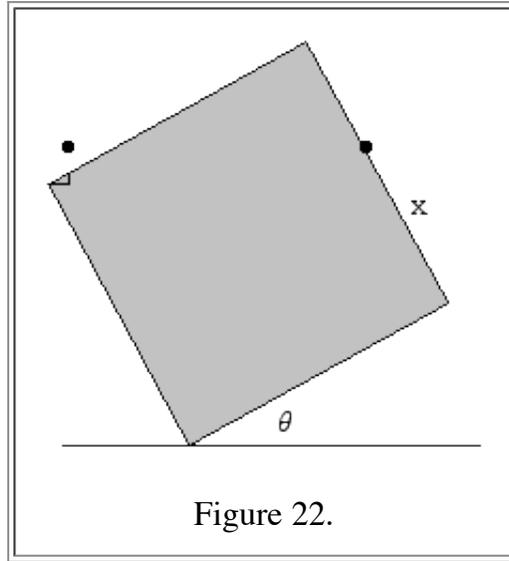
Figure 22.

Lemma 7. If a unit square has its center in the region $[0,1]^2$, does not contain either of the points (0,1) or (1,1), and is entirely above the x-axis, then the square covers either the point $(0,\sqrt{2}-1/2)$ or the point $(1,\sqrt{2}-1/2)$.

Proof: Lemma 2 shows that the center of the square cannot have y-coordinate less than or equal to $\sqrt{2}-1/2$ without covering one of these points. Lemma 4 shows that the center of the square cannot have y-coordinate more than $\sqrt{2}-1/2$ without covering one of these points. ∎

5 Lower Bounds

To show that $s(n) \geq k$, we will modify a method used by Walter Stromquist [13]. We will find a set P of (n-1) points in a square S of side k so that any unit square in S contains an element of P (possibly on its boundary). Shrinking these by a factor of (1-ε/k) gives a set P' of (n-1) points in a square S' of side (k-ε) so that any unit square in S' contains an element in P' in its interior. Therefore no more than (n-1) non-overlapping squares can be packed into a square of side (k-ε), and $s(n) > k-\varepsilon$. Since this is true for all ε>0, we must have $s(n) \geq k$.

We call P a set of *unavoidable* points in S. We now prove lower bounds on s(n) by showing that certain sets of points are unavoidable.

Theorem 1. s(2)=s(3)=2.

Proof: Consider a unit square u in $[0,2]^2$. Since the center of u is either in $[0,1]^2$ or $[0,1]\times[1,2]$ or $[1,2]\times[0,1]$ or $[1,2]^2$, Lemma 1 shows that u contains the point (1,1). That is, the set P={ (1,1) } is unavoidable in $[0,2]^2$ (see Figure 23). ∎

s(2)≥2
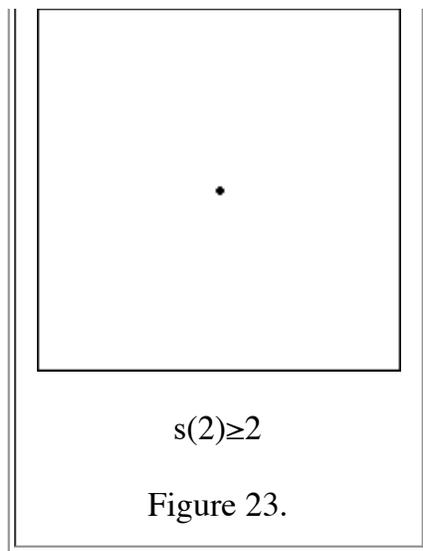
Figure 23.

Theorem 2. $s(5)=2+1/\sqrt{2}$.

Proof: The set $P=\{ (1,1), (1,1+1/\sqrt{2}), (1+1/\sqrt{2},1), (1+1/\sqrt{2},1+1/\sqrt{2}) \}$ is unavoidable in $[0,2+1/\sqrt{2}]^2$. This follows from Lemma 1 if the center of the square is in a corner, from Lemma 2 if it is near a sides, and from Lemma 3 if it is in a triangle (see Figure 24). ∎

s(5)≥2+1/√2
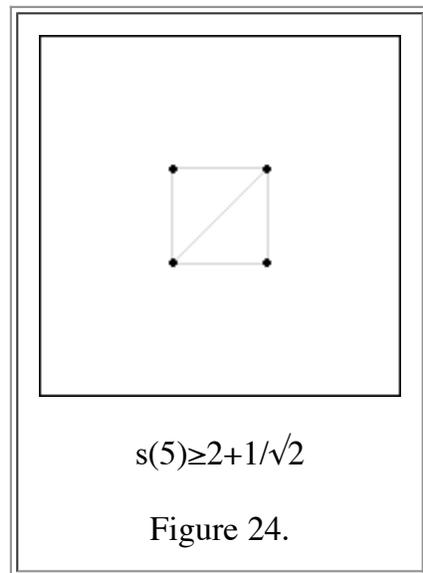
Figure 24.

Theorem 3. $s(8)=3$.

Proof: The set $P=\{ (.9,1), (1.5,1), (2.1,1), (1.5,1.5), (.9,2), (1.5,2), (2.1,2) \}$ is unavoidable in $[0,3]^2$ by Lemmas 1, 2, and 3 (see Figure 25). ∎

s(8)≥3
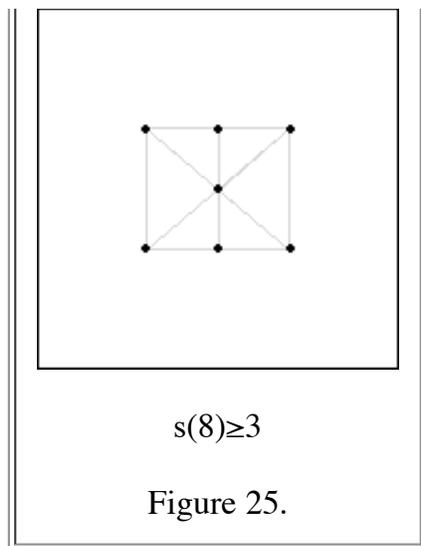
Figure 25.

Theorem 4. s(15)=4.

Proof: The set

P= { (1,1), (1.6,1), (2.4,1), (3,1), (1,1.8), (2,1.8), (3,1.8), (1,2.2), (2,2.2), (3,2.2), (1,3), (1.6,3), (2.4,3), (3,3) }

is unavoidable in $[0,4]^2$ by Lemmas 1, 2, 3, and 4 (see Figure 26). ∎

s(15)≥4
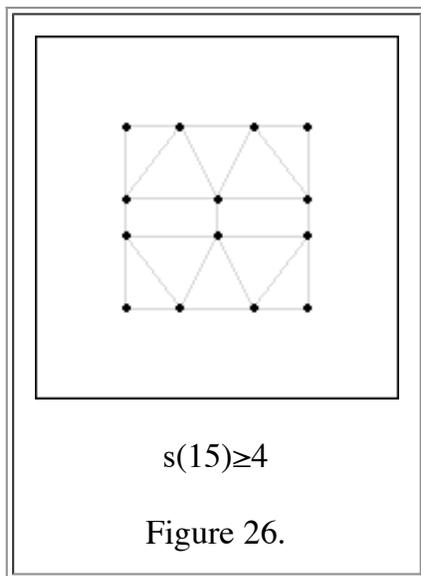
Figure 26.

Theorem 5. s(24)=5.

Proof: The set

P={ (1,1), (1.7,1), (2.5,1), (3.3,1), (4,1), (1,1.7), (2,1.7), (3,1.7), (4,1.7), (1,2.5), (1.5,2.5), (2.5,2.5), (3.5,2.5), (4,2.5), (1,3.3), (2,3.3), (3,3.3), (4,3.3), (1,4), (1.7,4), (2.5,4), (3.3,4), (4,4) }

is unavoidable in $[0,5]^2$ by Lemmas 1, 2, and 3 (see Figure 27). ∎

s(24)≥5
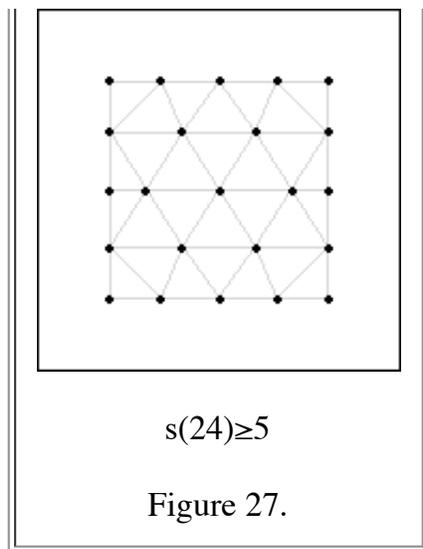
Figure 27.

Theorem 6. s(35)=6.

Proof: The set

P= { (1,.9), (2,.9), (3,.9), (4,.9), (5,.9), (1,1.725), (1.5,1.725), (2.5,1.725), (3.5,1.725), (4.5,1.725), (5,1.725), (1,2.55), (2,2.55), (3,2.55), (4,2.55), (5,2.55), (1,3.375), (1.5,3.375), (2.5,3.375), (3.5,3.375), (4.5,3.375), (5,3.375), (1,4.2), (2,4.2), (3,4.2), (4,4.2), (5,4.2), (1,5), (1.6,5), (2.4,5), (3,5), (3.6,5), (4.4,5), (5,5) }

is unavoidable in $[0,5]^2$ by Lemmas 1, 2, and 3 (see Figure 28). ■

s(35)≥6

Figure 28.
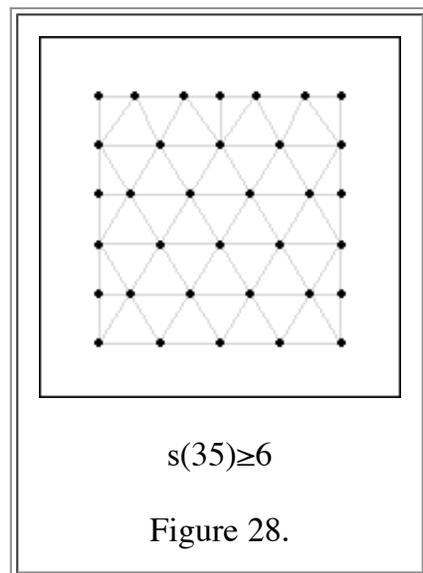
The proofs that s(7)=3 and s(14)=4 are a little harder. We find sets of points which are *almost unavoidable*, which force squares into certain positions. We use Lemmas 5, 6, and 7 to show that certain regions are covered, and find sets of unavoidable points for the rest of the square.

Theorem 7. s(7)=3.

Proof: If 7 unit squares are packed in a square of side 3-ε, at most 5 squares cover the 5 points in the almost unavoidable set shown in Figure 29. Therefore at least two squares have their centers in the regions containing question marks.
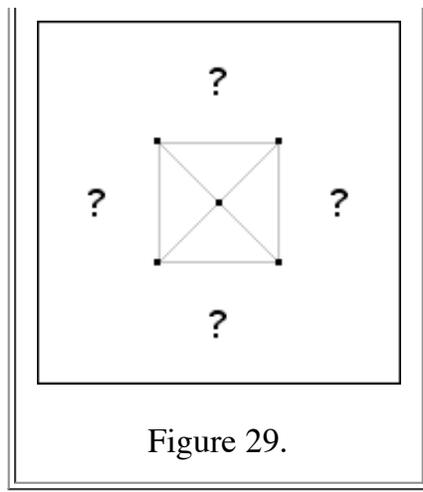
Figure 29.

There are 2 possible placements of these 2 squares up to rotation and reflection. Figure 30 shows these possibilities. The shaded trapezoids show points that must be covered by squares in those regions because of Lemmas 5, 6, and 7. Actually, we have drawn the trapezoids with a y value of 1/2 in Lemma 6 because this is the worst case in what follows. Each diagram shows a set of 3 additional unavoidable points. ■
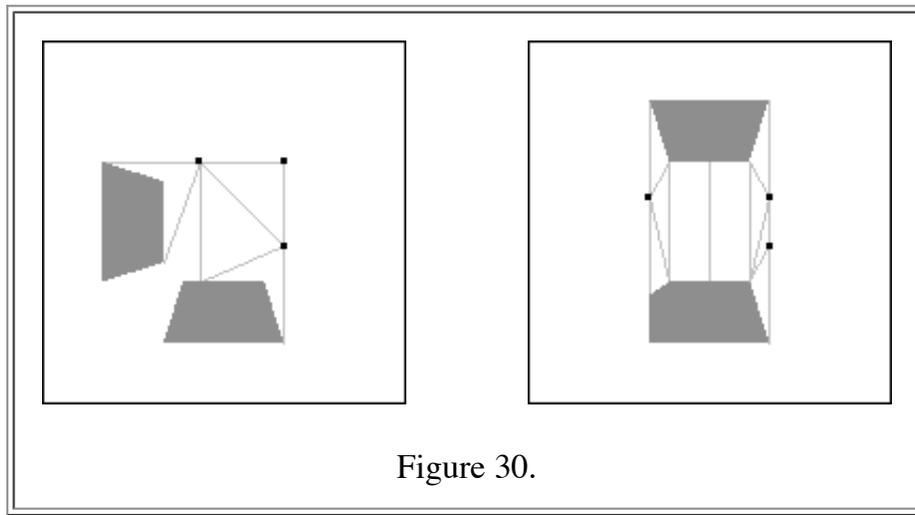
Figure 30.

Theorem 8. s(14)=4.

Proof: If 14 unit squares are packed in a square of side 4-ε, at most 12 squares cover the 12 points in the almost unavoidable set shown in Figure 31. Therefore at least two squares have their centers in the regions containing question marks.
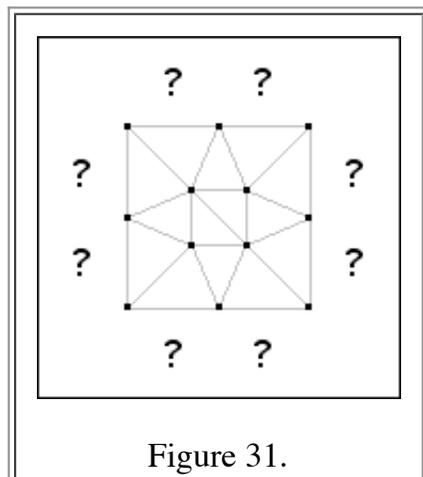
Figure 31.

There are 5 possible placements of these 2 squares up to rotation and reflection. Figure 32 shows the 5 possibilities.
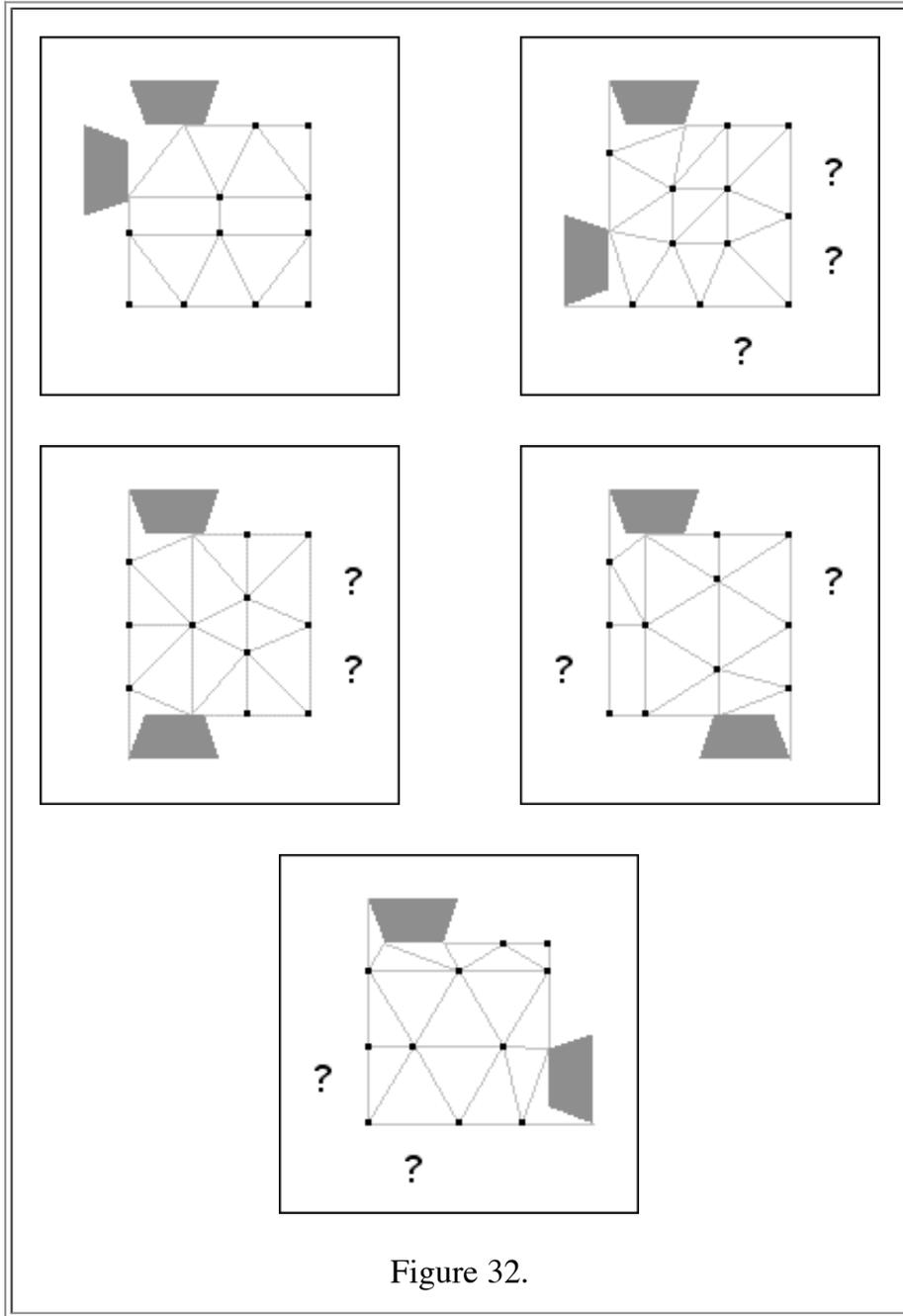
Figure 32.

Each diagram also shows a set of 11 additional unavoidable (or almost unavoidable) points. Some of these cases have additional cases, and these are shown in Figure 33.

Figure 33.
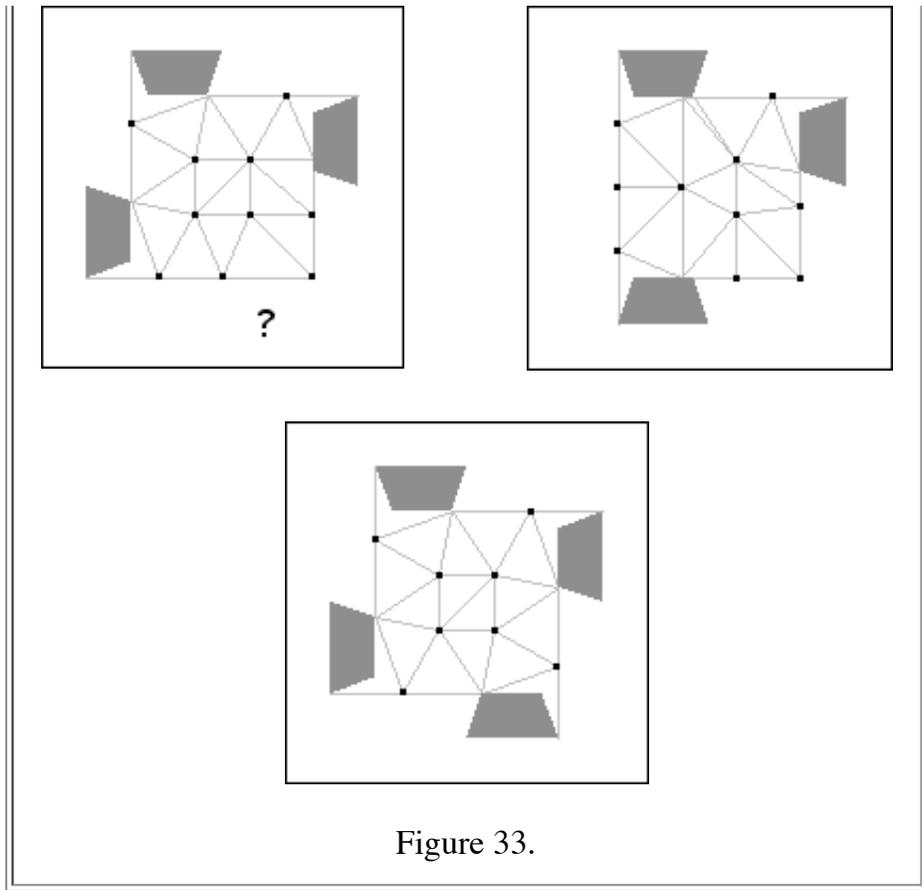
The other lower bounds known are probably not sharp. For example, Trevor Green has shown [8]:

Theorem 9. $s(n^2+1) \geq 2\sqrt{2} - 1 + (n(n-1)^2 + (n-1)\sqrt{(2n)}) / (n^2+1)$.

Theorem 10. $s(n^2+\lfloor n/2 \rfloor+1) \geq 2\sqrt{2} + 2(n-2)/\sqrt{5}$.

And Walter Stromquist has shown [18]:

Theorem 11. $s(11) \geq 2 + 4/\sqrt{5}$.

Unavoidable sets illustrating some of the lower bounds on s(n) are shown in Figure 34.

$s(17) \geq (40\sqrt{2}+19)/17$ $s(19) \geq 6\sqrt{2}-4$
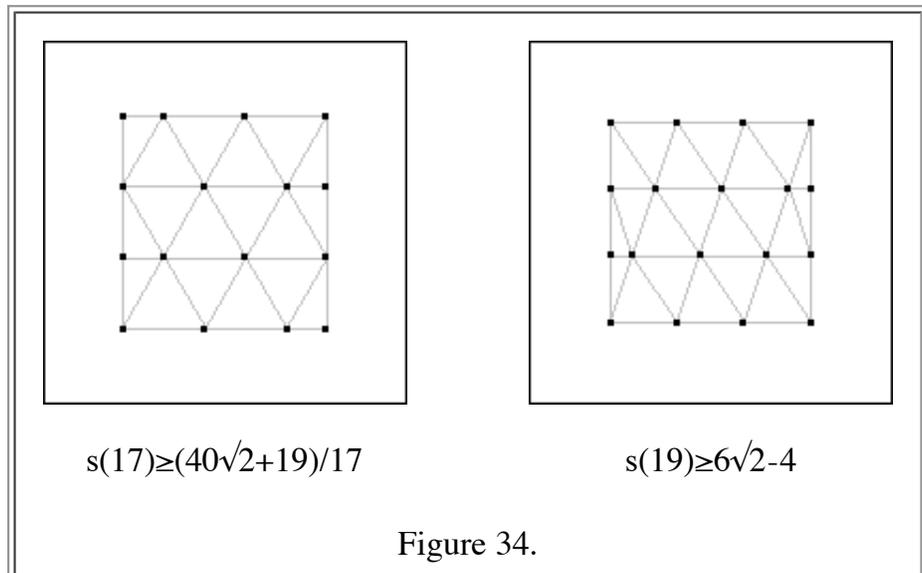
Figure 34.

Appendix

Table 1 contains the best known upper bounds on s(n) for n≤100. For each primitive packing, the Figure and the Author are given.

n	s(n)	Optimal?	Figure	Author
1	1	✓		
2-4	2	✓		
5	2+1/√2⪆2.7072	✓	Figure 1	Göbel
6-9	3	✓		
10	3+1/√2⪆3.7072	✓	Figure 1	Göbel
11	⪆3.8771		Figure 6	Trump
12-13	4			
14-16	4	✓		
17	⪆4.6756		Figure 12	Bidwell
18	7/2+1/2√7⪆4.8229		Figure 7	Hämäläinen
19	3+4/3√2⪆4.8857		Figure 4	Wainwright
20-22	5			
23-25	5	✓		
26	7/2+3/2√2⪆5.6214		Figure 9	Friedman
27	5+1/√2⪆5.7072		Figure 1	Göbel
28	3+2√2⪆5.8285		Figure 3	Göbel
29	⪆5.9344		Figure 12	Gensane/Ryckelynck
30-33	6			
34-36	6	✓		
37	⪆6.5987		Figure 16	Cantrell
38	6+1/√2⪆6.7072		Figure 1	Göbel
39	⪆6.8189		Figure 15	Cantrell
40	4+2√2⪆6.8285		Figure 3	Göbel
41	⪆6.9473		Figure 16	Cantrell
42-46	7			
47-49	7	✓		
50	⪆7.5987			
51	⪆7.7044		Figure 16	Hajba
52	7+1/√2⪆7.7072		Figure 1	Göbel
53	⪆7.8231		Figure 11	Cantrell

n	s(n)		Figure	Author
54	⪍7.8488		Figure 14	Cantrell
55	⪍7.9871		Figure 16	Cantrell
56-61	8			
62-64	8	✓		
65	5+5/√2⪍8.5356		Figure 3	Göbel
66	3+4√2⪍8.6569		Figure 5	Stenlund
67	8+1/√2⪍8.7072		Figure 1	Göbel
68	15/2+√7/2⪍8.8229		Figure 11	Cantrell
69	⪍8.8562		Figure 14	Cantrell
70	⪍8.9121		Figure 13	Cantrell
71	⪍8.9633		Figure 16	Cantrell
72-78	9			
79-81	9	✓		
82	6+5/√2⪍9.5356			
83	4+4√2⪍9.6569			
84	9+1/√2⪍9.7072		Figure 1	Göbel
85	11/2+3√2⪍9.7427		Figure 9	Friedman
86	17/2+√7/2⪍9.8229		Figure 15	Friedman
87	⪍9.8520		Figure 10	Cantrell
88	⪍9.9018		Figure 13	Cantrell
89	5+7/√2⪍9.9498		Figure 3	Stenlund
90-97	10			
98-100	10	✓		

Table 1. Best known upper bounds for s(n)

Table 2 contains the best known non-trivial lower bounds on s(n) for n≤85, along with the Author.

n	s(n)	Figure	Author
2-3	2	Figure 23	Göbel
5	2+1/√2⪍2.7071	Figure 24	Göbel
6	3		Kearney/Shiu
7	3	Figure 29	Friedman
8	3	Figure 25	Friedman
10	3+1/√2⪍3.7071		Stromquist
11-12	2+4/√5⪍3.7888		Stromquist
13	3.8437		Friedman
14	4	Figure 31	Friedman
15	4	Figure 26	Friedman

17-18	(40√2+19)/17≈4.4452	Figure 34	Green
19-20	6√2-4≈4.4852	Figure 34	Friedman
21	4.7438		Friedman
22	2√2+2≈4.8284		Green
23	5		Nagamochi
24	5	Figure 27	Friedman
26-27	2√2+(27+2√10)/13≈5.3918		Green
28-30	2√2+6/√5≈5.5117		Green
31	5.6415		Green
34	6		Nagamochi
35	6	Figure 28	Friedman
37-39	2√2+(113+10√3)/37≈6.3506		Green
40-41	2√2+8/√5≈6.4061		Green
47-48	7		Nagamochi
50-53	2√2+(101+3√14)/25≈7.3174		Green
62-63	8		Nagamochi
65-68	2√2+71/13≈8.2899		Green
79-80	9		Nagamochi
82-85	2√2+(288+12√3)/41≈9.2667		Green

Table 2. Best known lower bounds for s(n)

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